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Large Cardinals and Projective Determinacy

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To Carrie, Karl, Mom, and Dad.

Preface

This thesis comes out of two streams of motivation: purely mathematical considerations, and considerations relating to the history and philosophy of mathematics. Both aspects are reflected in the two very different, but closely interconnected halves of the present work.

The mathematical goal of this thesis is to provide a thorough and accessible exposition of a series of important results which, through the axiom of determinacy, link descriptive set theory—loosely speaking, the study of well-behaved subsets of the real line—to large cardinal axioms—that is, axioms which extend the "height" of the universe of sets. This portion of the thesis, Part II, culminates in a proof that $\mathbf{I}_0$ implies projective determinacy, which has never before appeared in print.

The non-mathematical aim of this thesis is to try to understand these results in their broader intellectual context. The historical portion of this thesis, Part I, examines the mathematical currents that coalesced into descriptive set theory in the late 19$^{\text{th}}$ and early 20$^{\text{th}}$ centuries. This historical development is illustrated through two case studies: first, the confluence of factors which led to the adoption of the infinite into mathematics as a legitimate object of study; and second, the French and Russian analysts who, at the turn of the 20$^{\text{th}}$ century, navigated between the revolutionary theory of sets first developed by Georg Cantor in the last decades of the 19$^{\text{th}}$ century and the traditions of mathematical analysis stretching back to the 17$^{\text{th}}$ century.

While very different in nature, it is the author's sincere hope that it will be evident how Part I and Part II are rooted in a common collection of interests and concerns and form a unified intellectual project.

Contents

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Part I

An Historical Approach

Chapter 1

Introduction

1.1 A role for history

In his 1962 *The Structure of Scientific Revolutions*, Thomas Kuhn (1922-1996) wrote that "History, if viewed as a repository for more than anecdote or chronology, could produce a decisive transformation in the image of science by which we are now possessed" [Kuh12, p. 1]. With these words, he ignited a debate that far outstripped his modest goal of producing an alternative to the linear, cumulative views then prevailing in the historiography of science. *The Structure of Scientific Revolutions* offered a host of analytical tools—incommensurability, paradigms and paradigm shift, the theory-ladenness of observation—that shook not only the history and philosophy of science, but also subjects as far afield as literary criticism and sociology [Bir13]. Within twenty-five years, over 650,000 copies of *Structure* had been printed, and *Structure* had become one of the most cited intellectual works in history [Kuh12, p. xxxvii]. Kuhn had ferried, almost overnight, the history of science from a young, niche field to one of the centerpieces of 20^{th} century academics.

But, while Kuhn's investigations prompted a broad scholarship in the history of physics, chemistry, and other natural sciences, considerably less has been devoted to studying the dynamics of change in mathematics. And yet, mathematics, is, at first blush, no less revolutionary: witness the turbulent rise of calculus in the 18^{th} century, or the dramatic work of Alexander Grothendiek (1928-2014), which brought about a fundamental reconception of vast tracts of 20^{th} century mathematics.

The inspiration for this essay is a shift that is less well-known, though certainly no less enigmatic. In 1962, the same year Kuhn published *Structure*, an insurrection was beginning in set theory. The seed of the change was a new axiom, the *axiom of determinacy* (**AD**). The axiom itself admits of a relatively simple description: whenever two players play a certain infinitely long game, one of the players has a strategy guaranteed to beat her opponent. For the finite analogue of this game, the assertion of **AD** amounts to a logical tautology [Lar12, p. 457]. But in the realm of the infinite, **AD** took on a very different character. Not only did it conflict with *Zermelo-Fraenkel set theory with the axiom of choice* (**ZFC**)—an axiom system which has, for close to one hundred years, served as the *de facto* standard for rigor[1]

[1] This assertion—namely, that mathematicians, when they think of it at all, think of **ZFC** as a sort of universal measuring stick for consistency or existence—is not to be confused with the subtly different *ontological* assertion that mathematicians think that mathematical objects are actually various kinds of sets.

in mathematics [Mad11, pp. 33-4]—but moreover **AD** seemed to lack a certain self-evidence which is often cited in support of the truth of **ZFC** [Eas08, p. 383]. Jan Mycielski (1932-) and Władysław Hugo Dionizy Steinhaus (1887-1972), the first mathematicians to consider **AD**, however, put these issues aside:

> "It is not the purpose of this paper to depreciate the classical mathematics with its fundamental 'absolute' intuitions on the universum of sets ... but only to propose another theory which seems very interesting although its consistency is problematic. Our axiom can be considered as a restriction of the classical notion of a set ... which reflect[s] some physical intuitions which are not fulfilled by the classical sets" [Lar12, qtd., p. 467].

But despite their early reservations, within two decades, the axiom of determinacy had become a keystone of modern set theory [Lar12, p. 458]. The reason was a totally unforeseen connection between the infinite games and a suite of long-standing and profoundly intractable problems in descriptive set theory.[2] Developments in the theory of functions of a real variable, which sought to resolve the difficulties and limitations of 19$^{\text{th}}$ century theories of integration and differentiation, prompted a pioneering group of French analysts—Borel, Baire, and Lebesgue—to classify how well- or ill-behaved the functions they studied could be. These classifications reduced, in turn, to certain classifications of how well- or ill-behaved different subsets of the real line could be. The conceptual tools they developed—measure, Baire category, the Borel hierarchy, and so on—helped to tame an increasingly unruly zoo of functions.

But their intellectual heirs—the school of Russian analysts headed by Luzin—soon found that natural questions engendered by the research of Borel, Baire, and Lebesgue seemed to resist any sort of resolution. Were the projective sets measurable? Did they satisfy Cantor's continuum hypothesis? Did they possess the property of Baire? In a remarkably prophetic note to *Comptes Rendus* in 1925, Luzin declared that the answer, quite simply, neither would, nor could, ever be known [Luz25b]. Developments in the theory of formal logic would soon bear out his prediction: by 1938, Kurt Gödel (1906-1978) had shown that it was consistent with **ZFC** that there be unmeasurable sets of relatively low complexity [Kan94, p. 169]; by 1970, Robert M. Solovy (1938-) had established that it was consistent with **ZF**[3] that *every* set, no matter its complexity, have the property of Baire, be measurable, and satisfy the continuum hypothesis [Sol70].

It would seem, then, that the difficulties encountered by Luzin were unsolvable in an extremely strong sense: their difficulty did not stem from a lack of human ingenuity or knowledge, but rather was an intrinsic feature of the mathematical landscape. And yet, today, seemingly impossibly, one regularly encounters assertions of the following sort:

[2] In the words of Thomas J. Jech (1944-), "[d]escriptive set theory deals with sets of reals that are described in some simple way: sets that have a simple topological structure (e.g., continuous images of closed sets) or are definable in a simple way. The main theme is that questions that are difficult to answer if asked for arbitrary sets of reals, become much easier when asked for sets that have a simple description" [Jec03, p. 131].

[3] That is, **ZFC** without the axiom of choice.

> "[Recent theorems in set theory linking projective determinacy[4] with large cardinals[5]] brought to a close a chapter which began over 60 years earlier with the questions of Luzin... With these theorems one can assert that **PD** is both consistent and true" [Woo10, p. 505].

1.2 Legitimacy in mathematics

The philosophical issues[6] of the episode outlined above, while inseparable from and fundamental to the topic at hand, raise an essential *historical* question of equal importance. The paradoxical narrative arc of **PD**—a mathematical question is raised, it is thought to be unsolvable, it is shown to be unsolvable, and then it is solved—raises the following difficult dilemma: what constitutes a *mathematical* solution to a problem? Is this a notion which has been stable over time, or is it subject to change? Perhaps most succinctly, would Borel, Baire, Lebesgue, and Luzin see as legitimate—or even *mathematical*—the modern resolution to the problem of **PD**?

These questions are inextricably tied to a more Kuhnian matter: what is the nature of change in mathematics? Is mathematics a subject that develops linearly and cumulatively? That is, does mathematics, as Hermann Hankel (1839-1873) asserted, differ from "most sciences [in which] one generation tears down what another has built[?] ... In mathematics alone each generation builds a new storey to the old structure" [Cro75, qtd., p. 165]. Or does mathematics develop episodically, as Kuhn suggests, guided by paradigms, the earliest of which date to prehistory [Kuh12, p. 15]?

The question of whether and how mathematics undergoes revolutions has come to be known in the literature as the *Crowe-Dauben debate*. Participants fall into two main camps: those, who, as Crowe, insist that "[r]evolutions never occur in mathematics" [Cro75, qtd., p. 165]; and those who contend, with Dauben, that "revolutions can and *do* occur in the history of mathematics" [Dau92, p. 50]. Few would deny that mathematics today is a different field than it was fifty or one hundred years ago. Rather, the essence of the debate is the problem of legitimacy: while no astronomer today still believes the Ptolemaic model of the heavens, no mathematician today would accuse Euclid of Alexandria of being wrong.

They might, of course, accuse him of being right for the wrong reasons—after all, Euclidean standards of rigor, exacting as they may be, pale in comparison to modern standards. But, significantly, the *content* of Euclid's *Elements* is still generally taken to be true. The accepted reasons for its truth, and the construal of those truths, however, are today much different from what they were two millenia ago. In the chapters that follow, I defend the assertion that this is a *general* phenomenon: while mathematical content does not undergo significant change, the framework in

[4] Projective determinacy (**PD**) is the assertion that all *projective* sets are determined. A consequence of **PD** is that the answer to every one of Luzin's unsolvable questions becomes "yes."

[5] A large cardinal is a set so large that, from its perspective, the collection of sets below it "looks like" a whole universe for **ZFC**. By Gödel's incompleteness theorem, the existence of such sets is independent of **ZFC**.

[6] The philosophical issues raised by such a position are the subject of a considerable literature. For an accessible introduction to some of the philosophical positions taken by those modern set theorists who view technical developments as having resolved the questions of Luzin, see [Mad11]. An incomplete sampling of this literature includes [Koe06], [Koe09], [Woo11a], [Woo10], [Woo11d], [Woo11b], [Mad88a], and [Mad88b].

which that content is set *does*. In this regard, I follow Dunmore and Gray in their assertion that mathematics undergoes meta-theoretic and ontological revolutions [Dun92], [Gra92].

However, I also maintain that the question of mere occurrence of revolutions falls short of the mark. The heart of the matter is, rather, why *some* new mathematical techniques and concomitant meta-level views gain acceptance in the mathematical community, while others founder. As Crowe observes, "[m]any new mathematical concepts, even though logically acceptable, meet forceful resistance after their appearance and achieve acceptance only after an extended period of time" [Cro75, qtd., p. 162]. The reason, I posit, is that *all* mathematicians are, in some sense, counter-revolutionary. Mathematicians, throughout history, have frequently felt strongly that their field ought to be "de-philosophized": questions pertaining to anything except the actual practice of mathematics were deflated; proper matters for investigation were restricted to that narrow ambit of knowledge that, in the idiom of Shapin and Schaffer, might be called "matters of fact," i.e., that particular kind of knowledge which is not open to theoretical—or, in our case, extramathematical—dispute [SS85, pp. 22-4]. In practice, this meant carving out domains of application for mathematical tools in which they had demonstrable value for solving already established problems. The question of what rose to the level of valuable or legitimate mathematics was also determined in reference to history; that is, rather than attempt to determine what "good" mathematics is from first principles, most mathematicians *took* to be valuable those mathematical investigations which preceded their own. Revolutions are therefore in some sense covert: those mathematicians who brought about revolutionary change in mathematics did so under the auspices of the mathematical methodology they ultimately displaced.

I shall defend this claim through the lens of two related case studies in the intertwined histories of set theory and analysis. In Chapter 2, I take a broad view, examining the manner in which infinity became a proper part of mathematics. We range from ancient Greek mathematics, during which period the relationship between the infinite and mathematics was first considered, to the birth of set theory in the late 19$^{\text{th}}$ century. Of particular concern is the degree to which contradiction, generally taken as the measure of mathematical illegitimacy, played a surprisingly unimportant role in the subject's development. In Chapter 3, I investigate a single historical moment: the development of descriptive set theory, first by Borel, Baire, and Lebesgue in France, and later by Luzin and his students in Russia. Here, I make special note of the values which informed their research: mistrust of the tendencies of set theory which they found unduly philosophical or theological; a special concern for the applicability of set theoretic techniques to problems in analysis; and the privileging of mathematics that was "effective," i.e., that bore a resemblance to concrete mathematical activities, such as counting. Both case studies can be viewed as part of a single thread—indeed, one of the most fruitful threads of mathematical history—stretching back to antiquity: the problem of measuring size.

Chapter 2

From Curves, to Functions, to Sets: 5^{th} Century B.C.E. to 1894

The fundamental tools of descriptive set theory considered in Chapter 3—to wit, measure, the Baire property, the perfect set property, and the Borel hierarchy—are directed toward some of the oldest problems in mathematics. All relate to notions of size—broadly construed—and have roots embedded in mathematical history stretching from the Greeks to the end of the 19^{th} century. The core questions of descriptive set theory emerged as the result of a dialectic process whereby intuitive—or, more accurately, pre-theoretic—notions underwent re-articulation, stretching, fission, and specification. This process of "concept-stretching," as Lakatos terms it, is particularly critical as it relates to the development of the function concept [Lak15, p. 83]. As functions emerged and, eventually, eclipsed curves as the object of mathematical study *par excellence*, a profound connection was formed with the unassuming notion of a line and the more enigmatic notion of the infinite; their co-evolution set the stage for the beginning of descriptive set theory proper in the works of Georg Ferdinand Ludwig Philipp Cantor (1845-1918) in the last quarter of the 19^{th} century.

Implicit in these technical developments were enormous changes in mathematical values, particularly as they relate to what does and does not qualify as legitimate mathematics. In the lead up to descriptive set theory, the development of mathematics followed a dynamic quite in opposition to two of the advertised values of 20^{th} century mathematics, and, not coincidentally, Greek mathematics: certainty and rigor. Indeed, following Kitcher, I argue that the theme of mathematics in this period was an *insensitivity* to theoretical inconsistencies [Kit83, p. 230]. Put differently, mathematicians treated their subject not as an unassailable body of knowledge, but rather as an art having a certain set of techniques, which, though potentially self-destructive, were nevertheless applicable within a safe domain. This safe domain was demarcated not explicitly, but rather on the basis of a kind of tacit knowledge gleaned from actual mathematical practice. Mathematicians enlarged the safe domain only in response to the successful application of new techniques to already legitimate, pre-existing questions, and inconsistencies only rose to the level of genuine counter-examples (i.e., problems necessitating revision of foundational concepts) when they emerged within the safe domain. Thus, mathematicians sought in large part to shield themselves from "logicians, philosophers, and other cranks interfering in their work" by pointing to the success and stability of the safe domain in spite of its possibly murky underpinnings [Lak15, p. 144]. It was rather in response to internal, mathematical issues that analysis, the subject of this chapter, underwent its celebrated period of rigorization. Indeed, as Lakatos, I argue that the possibility of counter-examples served in many ways as the engine, rather than the brake, of mathematical growth.

2.1 Infinity from Antiquity to the Enlightenment

2.1.1 Aristotle and the actual and potential infinite

The history of infinity is, appropriately, a long one. In Western mathematics, one can trace the concept to the Greek *to apeiron*[1], meaning, among other things, "infinitely large" and "undefinable" [Wal10, p. 44]. The term was first used in Greek tragedy to indicate "garments or binds 'into which one is tangled past escape' " [Wal10, p. 44]. No doubt this suspicion stemmed in large part from the famous paradoxes of Zeno of Elea (c. 490-425 B.C.E.) [Kli72, p. 175]. His four paradoxes of motion challenged the idea that one can properly think of space as a collection of points or time as a collection of moments [Kli72, pp. 36-7]. Zeno proposed that, for instance, "everything when it occupies an equal space is at rest, and if that which is in locomotion is always in a now [i.e., a single moment of time]; the flying arrow is therefore motionless" [AriCEb, VI.8].

Aristotle (385-323 B.C.E.) in answering Zeno, claimed that his argument was invalid, "for time is not composed of indivisible nows any more than any other magnitude is composed of indivisibles" [AriCEb, VI.8]. That is, to take the bite out of Zeno's arguments, Aristotle proposed the following distinction: an assemblage of an infinite number of distinct things—moments since the earth's creation, "sides" of a circle, points on a line, etc.—which Aristotle terms an "actual infinity," cannot exist: "For the fact that the process of dividing never comes to an end ensures that this activity exists potentially, but not that the infinite exists separately" [AriCEa, VI]. That is, what is infinite is merely the the *possibility* of always finding a further point on a line or a later moment in time. Thus, the line is not merely made up of points, as one cannot *actually* separate all of the points from each other at the same time.

2.1.2 The method of exhaustion

Greek philosophers took as dim a view of the infinite in their mathematics as they did in their metaphysics, but the mathematical infinite proved more difficult to excise. After counting, perhaps the oldest recorded mathematical problems are geometric, fashioned to the needs of construction and surveying [Boy68, p. 7]. Chief among these was the determination of the areas of different shapes, a problem considered both in Egypt and Mesopotamia in some of the earliest extant mathematical documents [Boy68, pp. 20, 44]. While Thales of Miletus (c. 620-546 B.C.E.), the semi-mythical founder of Greek philosophy and mathematics, ostensibly learned mathematics from those two cultures, it was in response to contradictory results obtained from the two earlier cultures' methods that he introduced logical structure to Greek geometry [Boy68, pp. 53-55]. While the status of Thales's existence and

[1] τὸ ἄπειρον

contributions to mathematics are murky at best, the rigor he is reported to have introduced was very much a defining feature of the Greek tradition that followed.

One of the greatest challenges for Greek mathematicians was therefore the determination of the area of curved shapes. For Babylonian and Egyptian mathematicians, for whom calculation rather than demonstration seems to have been primary, the problem was not markedly more difficult than finding the area of a square: for example, the Egyptian scribe Ahmes (fl. c. 1800 B.C.E.), whose papyrus serves as the most important record of early Egyptian mathematics, took the areas of a square field with a side length of eight units to be roughly the same as the area of a circular field with a diameter of nine units [Boy68, p. 20].[2] The calculation involves a step that, in the Greek tradition, would have been illicit: the circle is treated as having the same area as an octagon constructed from a square with a side length of nine units.

While Greek demonstration proscribed such elisions, Ahmes's method nevertheless foreshadowed the eventual Greek treatment of the problem. For the Greeks, the general problem of calculating an area was that of *quadrature*, i.e., a polygon of size equal to the given figure [Kli72, p. 41].[3] Some Greek mathematicians, such as Antiphon the sophist (fl. c. 400 B.C.E.), are reported to have claimed that if one circumscribed around a circle a square, and in it an octagon, and in it a hexadecagon, and so on *ad infinitum*, then one would eventually arrive at a regular polygon with an infinite number of sides, each of which would lie evenly with the circumference of the circle [Arc02, p. cxlii]. The areas of circumscribed polygons were known to be equal to that of a triangle with height equal to the radius and base equal to the polygon's perimeter; hence, the area of the *circle* would be equal to a triangle with height equal to the radius and base equal to the circumference.[4]

But could one accurately relate the area of a circle to a triangle without invoking the disallowed actual infinite? The answer would come from Eudoxus of Cnidus's (408-355 B.C.E.) method of exhaustion, an indirect and complicated procedure in which one approximates curved figures and solids with simple polygons and polyhedra. This is the method Archimedes of Syracuse (287-212 B.C.E.) used several hundred years later in *Measurement of a Circle* to demonstrate the theorem in requisite rigor [Arc02, p. 91].[5]

That it took so long to solve the problem after it was posed is indicative of the difficulty of the method of exhaustion. In order to avoid any mention of the infinite, one reasons on the basis of a double impossibility. In Archimedes's case, one begins with the observation that a polygon *inscribed* in the circle has a lesser area, and a *circumscribed* polygon has a greater area; however, by increasing the number of sides, the relative difference between the area of the circle and the polygon can be made less than any given magnitude.[6] Moreover, one can independently prove that

[2] In modern notation, if $A = \pi r^2$, this value of π used is approximately 3.16.

[3] As the name indicates, the problem was in fact to construct a *square* of the same size. However, once any polygon has been constructed, it is not difficult by Euclidean means to effect the construction of a square of equal size.

[4] That is, as one would expect, in modern notation, $A = r(2\pi r) = \pi r^2$.

[5] It is worth noting that Archimedes does *not*, technically, find the quadrature of the circle since he does not show how to construct the side of the triangle equal to the circle's radius. This is why most of *Measurement of the Circle* is occupied by calculating a numerical approximation to π.

[6] The Greeks separated sharply between the notions of *number*, i.e., positive whole numbers, and *magnitude*, i.e., quantities like area or weight that could be compared.

the areas of the circumscribed polygons are all greater than that of the triangle, and the area of the inscribed polygons all less than that of the triangle. Therefore, if the area of the circle were less than that of the triangle, one could circumscribe a polygon such “that the spaces intercepted between it and the circle are together less than the excess of [the triangle] over the area of the circle” [Arc02, p. 92]. It follows, then, that the area of the polygon would be both *smaller* than that of the triangle, since it differed by less from the area of the circle, and *greater*, since it was circumscribed about the circle. If the area of the circle were greater than that of the triangle, one could do the opposite. The only remaining possibility is therefore that the circle and triangle have equal areas.

While leaving no doubt about the truth of its conclusions, Eudoxus's method of exhaustion suffered from serious drawbacks. It was unwieldy, and required a different “ingenious scheme” for each approximation argument; often, it effected only a partial solution, producing a polygon that could not actually be constructed [Kli72, p. 177]. Moreover, in the early centuries of the common era, the Greek mathematical tradition entered a precipitous decline [Kli72, pp. 177-181]. Beginning in the 7$^{\text{th}}$ century, Islamic scholars, drawing also on a familiarity with Indian mathematics, would preserve and advance much of the mathematical knowledge of the Hellenistic world, especially in the areas of algebra, trigonometry, and numerical calculation. Nevertheless, advances in the theory of quadratures would not come until many centuries later [Boy68, pp. 253-276].

2.1.3 Infinite methods in Renaissance Europe

The confluence of numerous factors led to a rekindling of interest in the problem of quadratures and the development of the calculus in the late 17$^{\text{th}}$ century, not all of which can be named here. One important factor was that when Greek mathematical knowledge was transmitted to Europe beginning in the 13$^{\text{th}}$ century from Islamic mathematicians, such as Mohammed ibn-Musa al-Khwarizmi (c. 780-850),[7] new calculational technology, such as Hindu-Arabic numerals—a system far superior to that used by the Greeks—came with it. The works of Nicolaus Copernicus (1473-1543) and Galileo Galilei (1564-1642) were additional factors that lead to renewed interest in mechanics, introducing a panoply of hitherto unconsidered curves,[8] and, in the works of Galileo, relations between varying quantities, the earliest adumbrations of functions [Kli72, p. 338]. René Descartes (1596-1650), in an appendix to *Discourse on the Method of Reasoning Well and Seeking Truth in the Sciences*,[9] had effected a synthesis between geometry and algebra which introduced an even greater variety of curves whose quadratures and tangents were not, and almost certainly *could* not, have been calculated using Greek methods.

Perhaps most importantly, the Greek's *horror infiniti* had persisted only as an ideal rather than a dogma of mathematics. In particular, it was one which a number of enterprising European mathematicians chose to ignore, perhaps encouraged by strides forward in mathematical understanding of mechanics, astronomy, and other

[7]It is from “al-Khwarizmi” that the modern English term “algorithm” derives. The word “algebra” derives from the name of his principal work, *The Compendious Book on Calculation by Completion and Balancing* (*Hisab al-jabr w'al-muqabala*).

[8]While the Greeks considered a few mechanical curves, such as the trisectrix, by and large such objects did not figure in the mathematics of that period [Boy68, p. 76].

[9]*Discours sur la méthode pour bien conduire sa raison et chercher la vérité dans les sciences.*

domains. Nicholas of Cusa (1401-1464), like Antiphon before him, thought the circle an infinite-sided polygon [Boy68, p. 305]. Nicole Orseme (1323-1382), performed an early velocity integration using his "latitude of forms," and considered some infinite series, proving for instance that the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots \tag{2.1}$$

is infinite [Boy68, p. 293]. Kepler extended the methods of Archimedes to further solids of revolution by breaking them into infinitely many volumetric elements and dispensing with the latter's strict rigor. Bonaventura Cavalieri (1598-1647), a student of Galileo, published *Indivisible Continuous Geometry*,[10] putting forth the argument that a figure, e.g., a parallelogram, is actually made up of an infinite number of infinitely small *atomic*, or indivisble, parallelograms. All these mathematicians "had need for Archimedean methods, being practical men, but they wished to avoid the logical niceties of the method of exhaustion" [Boy68, p. 359]. The time was ripe for an innovation.

2.2 Analysis in the 17^{th}, 18^{th}, and 19^{th} Centuries

2.2.1 Newton and Leibniz's discovery of the calculus

The great insight of Isaac Newton (1643-1727) and Gottfried Wilhelm von Leibniz (1646-1716) was merely a surprising relationship between some of the problems studied by their predecessors. The two mathematicians independently observed that many of the problems

> "reduce to these two Problems only, which I shall propose concerning a Space described by a local Motion ... I. *The Length of the Space described being continually ... given; to find the Velocity of the Motion at any Time proposed.* II. *The Velocity of the Motion being continually given; to find the Length of the Space described at any Time proposed*" [New36, p. 19].

Newton demonstrated general methods for converting fluents, or "flowing Quantit[ies]" expressed as an algebraic relation between two quantities, to fluxions, "the Velocities by which every Fluent is increased by its generating Motion" and observed that given a curve, fluent[11] corresponding to the area contained beneath it along some abcissa had as its fluxion the length of the corresponding ordinate [New36, p. 20]. That is, the problems of finding tangents and quadratures were *inverse*, and could be carried out algebraically. (In Leibniz's idiom, these two methods would come to be known as the *calculus differentialis* and *calculus integralis.*)

The advantages of the new method over the miscellany of techniques possessed by their predecessors were immediately obvious. Newton, in *The Method of Fluxions*, derives in three lines the area of a parabola, the subject of Archimedes's entire treatise *Quadrature of the Parabola* [New36, p. 87], [Arc02, pp. 233-252]. While there were some differences as far as *what*, exactly, Newton and Leibniz thought the

[10] *Geometria indivisibilibus continuorum.*

[11] In modern terminology, "fluent" corresponds to "function" and "fluxion" to derivative.

calculus consists, their methods were largely the same: namely, given to quantities x and y related by the equation to increment both by "indefinitely little accessions" $x+\dot{x}o$ and $y+\dot{y}o$, and then, substituting these back into the original equation, solving for $\dot{y}/\dot{x}$ by neglecting any terms of the form ox, oox, and so on.[12] In Newton's words, the "tediousness of deducing involved demonstrations *ad absurdum*, according to the manner of the ancient geometers" could be wholly avoided [Kit83, qtd., p. 239].

Both understood, of course, that their methods possessed limitations, and they almost immediately met with resistance on account of those shortcomings. John Colson (1680–1760), the translator[13] of *Method of Fluxions*, felt compelled to observe in the preface that

> "It should however be well considered by those Gentlemen that the great number of Examples they will find here, to which the Method of Fluxions may be apply'd, are so many vouchers for the the truth of the Principles, on which that Method is founded. For the Deductions are always conformable to what has been derived from other uncontroverted Principles, and therefore must be acknowledged as true" [New36, p. xi].

For instance, when using the method of fluxions, the quantity $o\dot{x}$ must be non-zero, since one divides by it in calculating a fluxion. At the same time, $o\dot{x}$ must be zero, since the same calculation requires $x + o\dot{x} = x$. Were $o\dot{x}$ and $o\dot{y}$, in the words of George Berkeley (1685-1753), author of *The Analyst: or A Discourse Addressed to an Infidel Mathematician*, merely "Ghosts of departed Quantities" [Ber34, p. 18]? Were, indeed, "these modern Analytics[14] ... not scientifical" [Ber34, p. 22]? Especially in the continental tradition, one could not use the calculus without coming face to face with "a Part of such infinitely small Quantity, that shall be still infinitely less than it, and consequently though multiply'd infinitely shall never equal the minutest finite Quantity," i.e., an actual infinity, which Berkeley avers "is, I suspect, an infinite Difficulty to any Man whatsoever" [Ber34, p. 3].

While Newton especially in, for instance, *Principia Mathematica*, attempted to set his method on solid footing, neither could have answered Berkeley's attacks on satisfactory metaphysical grounds.[15] Nevertheless, Newton and Leibniz

> "were able to show that they could obtain, systematically, the solutions to problems previously recognized as important, solutions which had been achieved in bits and pieces by their predecessors; that they could answer, in some fashion, questions which others had unsuccessfully attempted to answer; and that they could solve problems that 'nobody previously had dared attempt' " [Kit83, p. 230].

Infinity, without the insulating layer of the method of exhaustion, had entered mathematics. And despite Berkeley's warnings, these problems were—as part of a familiar pattern—put aside until *mathematics* made them salient again.

[12] That is, if $xx = y$, then $x^2 + 2o\dot{x} + oo\dot{x}\dot{x} = y + o\dot{y}$, so, subtracting $xx = y$, $2xo\dot{x} + oo\dot{x}\dot{x} = o\dot{y}$, i.e., $\dot{y}/\dot{x} = 2x + o\dot{x} = 2x$. Note that Newton did not write x^2, as is conventional today, but rather xx.

[13] The *Method of Fluxions* was originally written in Latin and was unpublished until after Newton's death.

[14] The term "analysis" is generally taken to refer to the more mature discipline into which the methods of Newton and Leibniz evolved in the 18^{th} century. For the purposes of this essay, "analysis" and "the calculus" are interchangeable.

[15] Berkeley wrote *The Analyst* several years after both Newton and Leibniz had died.

2.2.2 The development of the function concept in the 18^{th} century

The methods of the calculus heralded an important reversal in mathematical priorities. Functions gradually usurped the position of curves as the central object of mathematical study. The first use of the word "function" occurred in a 1683 manuscript of Leibniz describing any quantity, such as tangents or ordinates, that varied along a curve [Kli72, p. 340]. As we have seen, the concept was already implicit in Galileo Galilei's *Two World Systems* [Kli72, p. 338]. Newton had a hand in the change as well: the assimilation of curves with various kinds of motion, lent further importance to functions, culminating in Newton's identification of curves with the paths of moving bodies [Kli72, p. 339]. Ultimately, it was the reduction of so many other kinds of problems to problems of functions that cemented their central role.

What functions precisely *were* was a matter of some difficulty, however. Leonard Euler (1707-1783), in his *Introduction to Analysis of the Infinite*,[16] defined a function to be "any analytic expression whatsoever made up from that variable quantity and from numbers or constant quantities," i.e., polynomials, transcendental functions, and power series [Boy68, qtd., p. 495]. Every function was, in modern terminology, continuous, differentiable, and anlytic, except possibly with small exceptions like the value $x = 0$ for $f(x) = 1/x$ [Kli72, p. 949]. This was a result, no doubt, of what mathematicians associated functions to: the equation, or expression, defining it via powers, fractions, sums, logarithms, and so on; and its power series.

The inseparability of a function from an equation expressing it and its power series stemmed from the indispensability of these two tools to the analytic methods of the time. The application of algebra to curves using the method of fluxions *required* a corresponding equation, and Newton relied heavily on the general binomial theorem to solve many of the more vexing problems he encountered [Kit83, p. 333]. Euler, for his part, thought that one could develop any function into power series, for "if anyone doubts that every function can be so expanded, then the doubt will be set aside by actually expanding the functions" [Kli72, qtd., p. 405]. Part of the justification came, moreover, from the enormous utility of thinking of functions as power series—using this insight, Euler, perhaps the most prolific mathematician ever to have lived, almost single-handedly solved many of the most important problems in 18^{th} century analysis, such as the Basel problem [Boy68, p. 496]. His techniques were so successful that his discoveries still form the backbone of much of college mathematics today [Boy68, p. 493]. So powerful was the connection between a function and its power series that it even provided a way for mathematicians to work productively with such obviously divergent series as

$$\sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + \dots, \tag{2.2}$$

taking the sum to equal -1 on the basis that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots \tag{2.3}$$

[16] *Introductio Analysin Infinitorum.*

In Euler's words, "since divergent series have no sum in the proper sense of the word, no inconvenience can arise from this new terminology. Finally, by means of this definition, we can preserve the utility of divergent series and defend their use from all objections" [Lis00, qtd., p. 259].

Despite these successes, a certain amount of doubt nevertheless pervaded analysis. In 1784, the Berlin Academy offered a prize to anyone who could clarify the foundations of the calculus [Lau00, p. 181]. It is perhaps telling that the prize winning memoir of Simon L'Huilier (1750-1840) foreshadowed the standards of rigor prevalent in the 19^{th} century, but was largely forgotten [Kit83, pp. 252-3]. His work was superseded by the sincere suggestion by Joseph-Louis Lagrange (1736-1813), in his influential textbook, that the difficulties of integrals and derivatives could all be avoided simply by observing that any function could be written in the form

$$f(x+h) = f(x) + ph + qh^2 + rh^3 + \dots \tag{2.4}$$

whence the derivative at x is merely p [Kli72, p. 431].[17]

2.2.3 The rigorization of the calculus

In spite of the foregoing arguments for a general *indifference* toward theoretical inconsistencies, a new current began to take hold in analysis around the turn of the 19^{th} century: rigorization. One finds in the introduction to Augustin Louis Cauchy's (1789-1857) celebrated *Course in Analysis*,[18] that he has

> "sought to give them all the rigor which one demands from geometry, so that one need never rely on arguments drawn from the generality of algebra. Arguments of this kind, although they are commonly accepted ... may be considered, it seems to me, only as examples serving to introduce the truth some of the time, but which are not in harmony with the exactness so vaunted in the mathematical sciences" [Cau09, pp. 1-2].

What prompted the change in attitude? The evolving needs of mathematical physics provided a strong impetus. The problem of modeling a vibrating string had attracted the attention of Johann Bernoulli (1667-1748) already in 1727 [Lau00, p. 180]. As understanding of the vibrating string problem grew, it became clear that Eulerian (or Lagrangian) functions, which had a determinate power series, would not suffice to solve the problem: one easily imagines, as Euler did, drawing a string into such a shape that different equations—that is, functions—obtained on different domains; these functions Euler called "discontinuous,"[19] writing to d'Alembert that "considering such functions as are subject to no law of continuity opens to us a wholly new range of analysis" [Boy68, qtd., pp. 506-7]. Daniel Bernoulli offered

$$u = \sum b_n \sin nx \cos nct \tag{2.5}$$

as a general solution to the vibrating string problem. So long as the sum is finite, though—as it was for Daniel Bernoulli—one cannot even capture Euler's "discontinuous" functions [Lau00, p. 180].

[17] To add to the irony, Lagrange actually faulted Newton for not realizing this.

[18] *Cours d'Analyse*.

[19] In modern terminology, the equivalent concept is that of *piece-wise analytic* functions.

However, a challenge would come in Jean Baptiste Joseph Fourier's (1768-1830) 1822 *The Analytical Theory of Heat.*[20] Fourier, in order to study the propagation of heat through a body, had conjectured that one can associate to "any function whatever" a trigonometric series converging to it. Echoing Euler, Fourier, first by analyzing functions in terms of their power series, arrived at a "very remarkable formula":

$$\begin{aligned}\frac{1}{2}\varphi(x) = \sin x \int \sin x\varphi(x)dx + \sin 2x \int \sin 2x\varphi(x)dx + \&c \\ + \sin ix \int \sin ix\varphi(x)dx + \&c \qquad (2.6)\end{aligned}$$

which, Fourier notes, "is important, because it shews how even entirely arbitrary functions may be developed in series of sines of multiple arcs ... [as] it is easy to represent the value of any integral term" [Fou78, p. 186].[21]

In particular, Fourier himself makes in essential use of the following infinite sum

$$\sin(x) - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \frac{1}{4}\sin(4x)\cdots \qquad (2.7)$$

which equals the function $f(x) = x/2$ between $x = -\pi$ and $x = \pi$, and repeats periodically elsewhere. Fourier, no doubt influenced by the connection to vibrating strings, asserted that at the points of discontinuity, vertical segments joined the function into an unbroken whole [Fou78, p. 144].

In addition to resolving outstanding problems in mechanics and the theory of heat, Fourier's techniques had immediate mathematical applications, offering, for instance, direct computations of the sums of many infinite series [Fou78, p. 147]. But these functions also stood in opposition to the 18$^{\text{th}}$ century notion of function: they defied representation by power series, and challenged the tacit assumption that functions were governed by the equations used to express them. Equation 2.7, for instance, is an expression equal to $x/2$ on the domain $(-\pi, \pi)$ and $x/2 - \pi$ on the domain $(\pi, 3\pi)$, i.e., the essential parity between functions and expressions had been broken.

It became *mathematically* necessary, therefore, to analyze what, precisely, was going wrong. Kithcer argues that

> "It is a gross caricature to suppose that Cauchy's work was motivated by a long-standing perception that mathematics had lapsed from high epistemological ideals... [Rather,] Cauchy recognizes the importance of infinite series representations of functions to questions in real analysis which he hopes to pursue, and that he recognizes that available algebraic techniques for manipulating infinite series expressions sometimes lead to false conclusions" [Kit83, p. 248].

[20] *Théorie Analytique de la Chaleur.*

[21] That is, Fourier, perhaps channeling the spirit of Euler, gives a proof which is based on the assumption that the function in question is analytic. From it, he derives Equation 2.6, which makes sense irrespective of whether the function $\varphi(x)$ is analytic. The conclusion that the formula holds for all functions is supported on the basis that for any *specific* function, one can still calculate the series. Thus, as in the case of Euler, the conclusion is based in part on the fact that the notion of an arbitrary function is still sufficiently restricted that no counter-example can be invented.

It is for this reasons that Cauchy makes explicit the once tacit notions that seemed to lead to trouble in the formerly safe domain: convergence, limit, derivative, integral, and continuity. Continuity, for instance, was determined in the following way:

> "the function $f(x)$ is a continuous function of x between the assigned limits if, for each value of x between these limits, the numerical value of the difference
> $$f(x+\alpha) - f(x)$$
> decreases indefinitely with the numerical value of α" [Cau09, p. 26].

In this framework, it was possible to state the criteria for the calculation of a Fourier series—integrability chief among others—and to *prove* or *disprove*, rather than merely assume, that series's convergence to the original function. Nevertheless, Cauchy's efforts were not entirely successful. In *Course*, Cauchy proves that

> "When the various terms of series (1) [viz., u_0, u_1, u_2, $\ldots$, u_n, u_{n+1}, $\ldots$] are functions of the same variable x, continuous with respect to this variable in the neighborhood of a particular value for which the series converges, the sum s of the series is also a continuous function of x in the neighborhood of this particular value" [Cau09, p. 90],

i.e., that the sum of a series of continuous functions is continuous.

The theorem had an undesirable quality, namely, "some exceptions," as observed by Niels Heinrik Abel (1802-1829) [Lak15, qtd. p. 133].[22] Abel complained of

> "the tremendous obscurity which one unquestionably finds in analysis... There are very few theorems in advanced analysis which have been demonstrated in a logically tenable manner. Everywhere one finds this miserable way of concluding from the special to the general and it is extremely peculiar that such a procedure has led to so few of the so-called paradoxes" [Kli72, qtd., p. 947].

But Abel's solution to the "tremendous obscurity"—namely, to restrict "function" once again to functions with power series, with which "in analysis one is largely concerned"—was merely a "monster barring": the paradoxes had already invaded the safe domain of mathematical practice as genuine counterexamples [Lak15, p. 133]. The path forward lay in reimagining the underpinnings of analysis. The spatial and temporal intuitions of the previous century would have to give way to focus on the real line itself.

2.2.4 Arbitrary functions and point sets

The works of two individuals in this period bear a particularly strong importance for what follows: Johann Peter Gustav Lejeune Dirichlet (1805-1859) and Georg Friedrich Bernhard Riemann (1826-1866). Through their work, functions would become, as implicit already in Fourier's *Theory*, arbitrary in the modern sense, and points of discontinuity as obstacles to integration and the convergence of Fourier series would become an important object of study in their own right.

[22]As Lakatos observes, one could have already noticed this from Fourier's Equation 2.7 [Lak15, p. 128].

In 1823, Cauchy had attempted to prove that the Fourier series of a function converged [Boy68, p. 605]. However, as Dirichlet discovered, the proof contained an error: Cauchy had assumed that if $\sum_n r_n$ is a convergent series and $\lim_{n\to\infty} s_n/r_n = 1$, then s_n converges also [Dau90, p. 7]. For instance, this property holds for the two series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \tag{2.8}$$

and

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \left(1 + \frac{(-1)^n}{\sqrt{n}}\right) \tag{2.9}$$

but Series 2.8 converges, while Series 2.9 does not.

In a series of papers beginning in 1828, Dirichlet, proceeding with a more careful analysis, determined that for functions with a finite number of minima and maxima and a finite number of discontinuities, the Fourier series converged to the original function, except at the discontinuities, where it converged to the average of the left hand and right hand limits [Boy68, p. 600]. His analysis began, however, with an analysis of the function concept generally. Confronted with the pathologies that had become increasingly important, Dirichlet conceived of a function as merely a determined value of y for each determined value of x, without "prescribing a common law" [Boy68, p. 600], [Mon72, qtd., pp. 62-3].

In fact, Dirichlet later determined that he could show the same result for a function with an infinite number of maxima, minima, and discontinuities: so long as the maxima, minima, and discontinuities were arranged away from 0 and containable in intervals of total length tending to zero,[23] it followed as corollary of his earlier results [Dau90, p. 10]. While Dirichlet had very successfully identified a host of *sufficient* conditions for the convergence of a Fourier series, the question of necessity remained. It is this thread that Riemann picked up in his 1854 *Habilitationsschrift On the Representability of a Function by Means of Trigononmetric Series.*[24]

Riemann begins by observing two things: first, that Fourier series were "'much used in mathematical physics to represent arbitrary functions"; and second, in the spirit of his forebears, that, for Fourier series, "in every individual case one became easily convinced that [they] converge[] toward the function's value" [Rie07, p. 997]. Nevertheless, Fourier series were then also "successfully being applied in an area of pure mathematics, number theory, and it is precisely here that those functions not investigated by Dirichlet for their representability by a trigonometric series are of importance" [Rie07, p. 1001]. Riemann therefore undertook to establish the *necessary* conditions for the convergence of a trigonometric series, considering for the first time the possibility of a series *not* generated by a function.[25] Doing so for the broader range of functions now used in mathematics required Riemann to redefine the integral, since the definition given by Cauchy in *Summary of the Lessons given at the Ecole Polytechnique on the Infinitesimal Calculus*[26] depended heavily upon

[23]This is, of course, an important early version of *measure*, a topic to which we shall turn in Chapter 3.

[24]*Ueber die Darstellbarkeit eine Function durch einer trigonometrische Reihe.*

[25]That is, Riemann considered the bare sum $b_0/2 + \sum_{n=0}^{\infty} a_n \sin(nx) + b_n \cos(nx)$ without regard to some function f from which the coefficients were generated. He was the first, therefore, to consider the possibility of a *Trigonometric* series, rather than a *Fourier* series.

[26]*Résumé des leçons données à l'École Polytechnique sur le calcul infinitesimal*

the function being monotonic.

Both Dirichlet and Riemann considered functions whose points of discontinuity exhibited a variety of pathological behaviors, such as being discontinuous at every rational number [Rie07, p. 1004]. This theme was further developed by Hermann Hankel (1839-1873) and Rudolf Otto Sigismund Lipschitz (1832-1903), who considered, for instance, functions with discontinuities on sets that, in modern parlance, are called "nowhere dense" [Dau90, p. 20].[27] Hankel attempted to construct functions whose sets of discontinuities had extremely complicated structures, explicitly identifying the notion of a point set being dense in an interval [Dau90, p. 24]. This effort was attendant, however, to his goal of rigorously delimiting the "legitimate functions" which possessed no pathologies [Dau90, p. 26-7]. While their work advanced analysis greatly, it remained to a later investigator to see how far one could plumb these new notions.

2.3 Georg Cantor and the Beginning of Set Theory

2.3.1 Derived sets and Cantor's discovery of the transfinite

Such was the mathematical world Georg Cantor inherited when he entered the University of Halle as a *privatdozent* in 1869 [Dau92, p. 58]. Born in 1845 in St. Petersburg, Cantor began his education in Germany in 1856, where his father hoped he would study to become an engineer [Wal10, p. 171]. Cantor's gift for mathematics was apparent from a young age, however, and he soon devoted himself to pure mathematics [Wal10, p. 171].

Cantor's early works considered problems in number theory. However, on the advice of Heinrich Eduard Heine (1821-1881), Cantor turned his attention in 1870 to one of the most important outstanding problems in trigonometric series: that of uniqueness of representations [Dau90, pp. 30-1]. It was still not known if two different trigonometric series could converge to the same function.

Between 1870 and 1872, Cantor provided a series of more and more general answers to this question [Dau92, p. 58]. Drawing on the work of Riemann before him, Cantor noted that to a trigonometric series $f(x) = b_0 + a_1 \sin x + b_1 \cos x + a_2 \sin 2x + b_2 \sin 2x + \ldots$ with bounded coefficients, one could associate the function[28]

$$F(x) = C_0 \frac{x \cdot x}{2} - C_1 - \ldots - \frac{C_n}{n \cdot n} \cdots \tag{2.10}$$

obtained by formally integrating each summand $C_k = a_k \sin kx + b_k \cos kx$ [Dau90, p. 33]. The function $F(x)$, as earlier observed by Riemann in his *Habilitationsschrift*, had convergence properties superior to those of the original function $f(x)$. In particular, in the case that $f(x) = 0$ identically on the interval $[-\pi, \pi]$, Cantor, using a lemma communicated to him by Hermann Amandus Schwarz (1843-1921), was able to show that $F(x)$ is linear, and hence that

$$-c' = C_1 + \frac{C_2}{2^2} + \ldots + \frac{C_n}{n^2} + R_n \tag{2.11}$$

[27] The nowhere dense sets will play an important role in Chapter 3.

[28] One assumes, of course, that $f(x)$ is partial and takes values only where the series is convergent.

where R_n was a quantity which Cantor had earlier proved was a function in x tending uniformly to 0 as n tended to infinty [Ash89, p. 883], [Dau90, p. 33]. Therefore, multiplying through by $\cos nx$ or $\sin nx$ for appropriate n and integrating,[29] Cantor obtained that $a_n = b_n = 0$.

However, Cantor soon realized that his result could be extended to the case that $f(x)$ is not assumed to converge to 0 at $-\pi \leq x_1, \ldots, x_n \leq \pi$, since linearity held on $(-\pi, x_1)$, (x_1, x_2), ..., (x_n, π). The key was that the proof depended only upon the linearity of $F(x)$, and that linearity could be deduced on any *interval* of convergence. Therefore, by approximating the given interval from the inside by proper subintervals, one could deduce that $F(x)$ was linear on $(-\pi, x_1)$, (x_1, x_2), ..., (x_n, π); and hence, by continuity, on all of $[-\pi, \pi]$.

But Cantor did not stop there. By precisely the same logic, if $x_1, x_2, x_3, \ldots$ were an increasing sequence with limit point x_∞, and the convergence of $f(x)$ were not assumed at any of those points, then by the same methods, one could show first that $F(x)$ was linear on $[-\pi, x_\infty)$ and $(x_\infty, \pi]$, from which it immediately follows that $F(x)$ is linear on all of $[-\pi, \pi]$. One could then ask the same question, except with each of $x_1, x_2, \ldots$ *itself* a limit of some sequence $x_1^1, x_2^1, \ldots$, $x_1^2, x_2^2, \ldots$, and so on. The answer, of course, was the same.

For a set P of points, Cantor denoted by P' *its* limit points, by $P^{(2)}$ the set $(P')'$, and so on. The general theorem which Cantor had arrived at was as follows: if the set of points P at which $f(x)$ did not exist were such that $P^{(d)}$ were finite for some d, then the trigonometric series representation was unique. The innovation, however, was that by performing the operation on a *set*, rather than a number or function, Cantor had tacitly rendered the point set an independent mathematical object.

That was as far as Cantor deigned to go in his 1872 *On the Extension of a Theorem from the Theory of Trigonometric Series*,[30] but already his thoughts had turned to the next logical possibility: a limit point x_∞ where,

$$x_1, x_2, \ldots, x_n \notin P^{(n)} \quad \text{but} \quad x_{n+1}, x_{n+2}, \ldots \in P^{(n)}. \tag{2.12}$$

Clearly, the theorem ought to hold also in this case, since, as Cantor was already considering at that time, $P^{(\infty)} = \emptyset$ [Dau90, p. 49]. The question was, what sense could be given to the expression "$P^{(\infty)}$"? At the end of the same paper, Cantor enigmatically wrote that "[t]he number concept, in so far as it is developed here, carries within it the germ of a necessary and absolutely infinite extension" [Dau90, qtd., p. 44].

2.3.2 Ordinals, cardinals, and the continuum hypothesis

Over the course of the next two decades, Cantor would cultivate that germ into a theory that would help set the agenda for 20th century mathematics. Much of this would come from Cantor's probing of the number concept, spurred on by his discoveries in the theory of trigonometric series. Specifically, Cantor recognized two distinct ways the integers arise in mathematics: in iterating processes, and in

[29] The point is, because R_n decreases to zero uniformly, the integration is licit.

[30] *Uber die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen.*

counting sizes. Understanding these two aspects—the *ordinal* and *cardinal*[31]—would spawn a program of research that continues into the present.

In considering how to extend his arguments in trigonometric series, Cantor was soon led to the concept of a *well-ordering*, that is,

> "a well-defined set in which the elements are bound to one another by a determinate given succession such that (i) there is a *first* element of the set; (ii) every single element ... is followed by another determinate element; and (iii) for any desired finite or infinite succession of elements there exists an element which is *their immediate successor* in the succession (unless there is absolutely nothing in the succession following them)" [Can05a, §2.2].

The source of these three generating principles can be seen immediately in the iterations into the infinite Cantor had considered almost a decade earlier. The transfinite *ordinal numbers*, the objects representing a particular order type, provided a basis on which Cantor could make sense of the iteration into the transfinite. The least of these infinite numbers Cantor dubbed ω. The enigmatic $P^{(\infty)}$, the derived set which after $P, P', P^{(2)}, \ldots$, could now be viewed as $P^{(\omega)} = \bigcap_{n<\omega} P^{(n)}$. However, $P^{(\omega)}$ has its limit points, namely, $(P^{(\omega)})' = P^{(\omega+1)}$. Continuing on in this way, Cantor came to consider $\omega + \omega$, 3ω, ω^2, ω^ω, and so forth through a whole arithmetic hierarchy. The arithmetic was, to be sure, different from its finite analogue: for instance, in contrast to finite numbers, commutativity did not hold, e.g.,

$$P^{(1+\omega)} = (P')^{(\omega)} = P^{(\omega)} \neq (P^{(\omega)})' = P^{(\omega+1)}. \tag{2.13}$$

The difference, Cantor wrote,

> "between finite and infinite sets now turns out to be that a finite set presents the same *Anzahl* [i.e., ordinal number] for every succession which one can give its elements; in contrast, a set consisting of infinitely many elements will in general give rise to different Anzahlen, depending on the succession that one gives its elements" [Can05a, §2.3].

Around the same time, Cantor had come to consider a more general problem of size. In a letter to Julius Wilhelm Richard Dedekind (1831-1916) in 1873, Cantor wrote regarding a question of a "certain theoretical interest" which he could not answer, namely, can the totality of all real numbers be correlated with the totality of all natural numbers such that "to each individual of the one totality there corresponds one and only one of the other" [Can05c]? It was clear, if unobvious,

> "that (n) [i.e., the totality of natural numbers] can be correlated one-to-one not with [the totality of rational numbers], but with the more general
>
> $$(a_{n_1,n_2,\ldots,n_v})$$
>
> where $n_1, n_2, \ldots, n_v$ are unrestricted positive integer indices in arbitrary number v" [Can05c].

[31] In the past, the word "power," from Cantor's original German "Mächtigkeit," was prevalent.

Within a week, Cantor had made his astonishing discovery: no [Can05c]! Over the next several years, Cantor amassed many such results, culminating in his 1877 discovery that a surface, or a manifold, of n-dimensions, could be correlated one-to-one with the line [Can05b].

Having come to the shocking realization that there was not just one infinite, but rather *different* infinite sizes and orders, the difficulty of producing sets of different sizes prompted Cantor to conjecture in 1878 that "the number of linear manifolds that this principle of sorting [i.e., into cardinalties] gives rise to is two" [Ewa05, p. 879]. This, his continuum hypothesis, would monopolize his research from then on. Taking his cue from his earlier investigations in trigonometric series, Cantor would attack the problem from two angles: directly, through the study of point sets; and indirectly, through the study of the alephs, the cardinalities of sets of ordinal numbers [Hal84, p. 3].

The first of these would trace back to the roots of Cantor's set theory in the study of derived sets. In an 1884 paper, Cantor attempted to show that any subset of the real line P could be decomposed into the union of R and S such that $S' = S$ and $P^{(\alpha)} = S$ for some countable ordinal α [Dauben, book 113]. Ivar Otto Bendixson (1861-1935), however, soon spotted an error: one had to assume that $P' \subset P$. The patched-up proof, now called the Cantor-Bendixson theorem, established the beginnings of a point set typology. Sets such that $P' = P$ would come to be known as "perfect" and those such that $P' \subset P$ "closed"; the attempt to understand the properties of these special sets, as in the corrected *Cantor-Bendixson theorem* supplied the initial impulse for descriptive set theory.

On the other hand, Cantor's second approach would give rise to what is generally considered his crowning achievement: the mature theory of well-orderings and cardinalities. It did not take long for Cantor to realize that the set of all well-orderings that could be rearranged into the well-ordering ω [32] themselves formed a collection that could not be put in one-to-one correspondence with ω. Thus, the cardinalities of the well-orderings, which he termed "the alephs," could be themselves arranged in a well-ordering:

$$\aleph_0 < \aleph_1 < \aleph_2 < \ldots < \aleph_\omega < \aleph_{\omega+1} < \cdots < \aleph_{2\omega} < \cdots \aleph_{\omega^2} < \cdots \tag{2.14}$$

The similarlity between the impossibility of bijecting[33] ω with the real numbers and with ω_1 prompted Cantor to consider his continuum hypothesis from a new angle: was it the case that $2^{\aleph_0} = \aleph_1$? In his last major published works in 1895 and 1897, *Contribution to the Founding of the Theory of Transfinite Numbers*, Cantor developed in great detail their arithmetical properties. Nevertheless, he never succeeded in proving the continuum hypothesis.

2.3.3 An uncertain future

Cantor's two different lines of attack on the continuum hypothesis would also spawn two different set theoretic traditions, one of which we shall examine in detail in Chapter 3. The core differences between these two traditions are revealed in the

[32] The countable ordinals formed Cantor's "second number class"; "the first number class" denoted the finite integers.

[33] A bijections is a one-to-one correspondence between all of the elements of one set with all of the elements of another.

different ways that Cantor attempted to legitimize his set theory in late 19$^{\text{th}}$ century mathematics.

The degree to which set theory had broken with the mainstream mathematical methodology was no doubt obvious to Cantor. In the preface to his 1883 *Foundations of a General Theory of Manifolds*,[34] he writes that his work is aimed both at "philosophers who have followed the developments in mathematics up to the most recent times, and for mathematicians who are familiar with the most important results, ancient and modern, of philosophy" [Can05a]. That is, Cantor understood that while the development of analysis had forced mathematicians to come face to face with the infinite in some regards, such luminaries as Johann Carl Friedrich Gauss (1777-1855), still claimed without irony that "I protest above all against the use of an infinite quantity [*Grösse*] as a *completed* one, which in mathematics is never allowed" [Dau92, qtd., p. 60]. To make sense of even $\omega + 1$, completed (i.e., actual) infinities were wholly unavoidable.

Cantor also appreciated that, as has been true throughout mathematical history, few would make the leap for purely philosophical reasons. Thus, Cantor vaunted the advances his theory made possible in the theory of functions, citing, for instance, discoveries Magnus Gösta Mittag-Leffler (1846-1947) had made with its aid. In the same article, Cantor made explicit again and again the "fruitfulness" of his techniques [Ewald, 51, 59]. Even earlier, he had succeeded in using the theory of cardinalities to prove Liouville's theorem; however, he presented the result "as though [he] wanted it to appear that his proof was neither new nor particularly special, except for its generality and its verification, in a new way, of Liouville's work" [Dau90]. It was this vision of set theory which would be received—begrudgingly—by analysts in France, Russia, and Poland.

At the same time, Cantor evidently thought that set theory could not be justified on the basis of mathematical fecundity alone. He rejected the "religious dogma" of Aristotle, Epicurus, St. Thomas Aquinas, and Kant that the continuum is "an unanalysable concept, or as others express themselves, a pure *a priori* intuition which is scarcely susceptible to a determination through concepts" [Can05a, §10.3]. And indeed, Cantor's prediction that his set theory would become an important object of philosophical debate, on which he "shall [not] have said the last word" was correct [Can05a, Preface].

In private, Cantor harbored views totally anethema to the prevailing vision of mathematics. Cantor maintained not that the universe was divided between the finite and the infinite, but rather between the finite, a comprehensible infinite realm he termed the *transfinite*, and an *absolute*, of which he spoke in reverent, spiritual terms:

> "I have rigorously proven that there is absolutely no 'Genus Supremeum' of the actual infinite. What surpasses all that is finite and transfinite is no 'Genus'—it is the single completely individual unity in which everything is included, which includes the "Absolute" incomprehensible to human understanding" [Hal84, qtd., p. 13].

Cantor was, in this regard, a pseudo-idealist: the mind of God was an indispensible part of the theory of sets, the force that guaranteed the existence and objectivity of the ordinals, cardinals, and other transfinite objects [Hal84, p. 14].

[34] *Grundlagen einer allgemeinen Mannigfaltigkeitslehre.*

But even the protection of theology could not protect the full extent of Cantor's set theory. In a letter to Dedekind in 1899, Cantor writes of "inconsistent multiplicities"—assemblages which are, in some sense, too large to be sets:

> "We assign to equivalent 'sets' one and the same determinate power-*class* and non-equivalent sets to different classes. We consider the system
>
> *S of all thinkable classes a.*
>
> . . . I maintain that the fully-determinate well-defined system S is *not a 'set'* " [Can05d].[35]

Cantor maintained that "the *essence* of *mathematics* lies precisely in its *freedom*" [Can05a, §8.5]. But freedom evidently came at a price. Set theory was still young and, in the eyes of many, mathematically unproven and tainted by un-mathematical tendencies. The struggle to properly delimit the mathematics of set theory from the philosophy of set theory would become one of the most important developments in early 20$^{\text{th}}$ century mathematics. It is to that fight to establish the proper values of mathematics to which we now turn.

[35] Cantor's proof—which amounts essentially to the contradiction that there cannot be a set of cardinal number greater than that of S—is essentially the same as the paradox discovered several years earlier by Cesare Burali-Forti (1861-1931).

Chapter 3

Traditionalism: 1894 to 1925

Almost from its inception, Cantor's theory of sets fueled a battle between warring factions in the mathematical community. Many, like Leopold Kronecker (1823-1891)—first Cantor's teacher and later his fiercest critic—assailed his works as "humbug" devoid of meaning [Dau90, p. 134]. Kronecker at times went so far as to attempt to suppress the subversive content, which he saw as going irretrievably beyond the concrete realm of the theory of numbers on which he thought all of mathematics should be based [Dau90, p. 135]. Others, like Julius Wilhelm Richard Dedekind (1831-1916) and David Hilbert (1862-1943) supported the transfinite enterprise in its entirety and championed its founder.

However, a third faction soon emerged that would ultimately lend to Cantor's theory the mathematical legitimacy he sought. Ironically, that legitimacy would come at the expense of Cantor's set-theoretic pride: the higher transfinite universe. These mathematicians—Félix Edouard Justin Émile Borel (1871-1956), René-Louis Baire (1874-1932), Henri Léon Lebesgue (1875-1941), Nikolai Nikolaevich Luzin (1883-1950), and others—whom I have dubbed the traditionalists, found virtue in the successful application of set theory to established problems in mathematics, and proscribed its application beyond these domains. These mathematicians, who viewed themselves first and foremost as *analysts*, rather than set theorists, saw the particular potential of Cantor's theory to advance the study of functions of a real variable and applied it to questions of continuity, derivation, integration, and classification. While the rise of set theory, it was generally agreed, had unavoidably introduced matters of ontology and epistemology relating to the infinite into mathematics, one could take a position on the issues that was thoroughly *mathematical*: the transfinite was to be grounded in more familiar mathematical objects, like point sets and functions, and oriented toward objects and problems that could actually be given, defined, or encountered;[1] consequently, set theory itself was supportable as far as it had genuine mathematical application, but not further. Said in another way, these mathematicians saw themselves as part of a tradition which Cantor had *augmented*, but not *usurped*. Set theory, after emerging from the mathematical mainstream of point sets and trigonometric series had, in their view, gone astray; they took as their task to turn it back onto its analytical roots. Three values in

[1]Drawing a bright line between, for instance, functions that are given and functions that are not is a very difficult philosophical problem, one to which I moreover do not think the traditionalists had an entirely satisfactory solution. What is important, however, is not whether or not they succeeded in making this distinction, but rather the fact that they tried to do so on the basis of what one might call "familiar mathematics."

particular informed this project: concern for "effectiveness," interest in application, and mistrust of any set theory beyond what proved absolutely necessary to their analytic goals.

From the works of the traditionalists arose the three fundamental "regularity properties" of descriptive set theory—measure, the Baire property, and the perfect set property—as well as the modern foundations of the fields of real analysis, probability theory, topology, and much more. Yet, even as they made great strides in advancing mathematical understanding, they soon came to grapple with problems at the limits of their methods. Despite their best effort to conform to their received standard of mathematical propriety, the traditionalists were unable to dispel a totally novel and undesirable phenomenon: unsolvability. Their excursions into set theory would, in this way, rehearse some of the central themes of later 20$^{\text{th}}$ century set theory.

3.1 The French School: 1895-1905

Perhaps the defining moment in the establishment of set theory in the 20$^{\text{th}}$ century occurred at the second International Congress of Mathematicians, held in Paris in 1900. David Hilbert (1862-1943), the leading German mathematician of the day, and one of mathematics's last universalists, delivered the conference's opening address. In what would become the most famous speech in mathematical history, Hilbert posed to his audience a list of ten outstanding problems for 20$^{\text{th}}$ century mathematics [GK09, p. 33]. The first among them related to "[t]he most suggestive and notable achievements of the last century in this field" [Hil02, p. 445]: Cantor's continuum hypothesis.

Set theory had been present in France for nearly a decade at that point, though most Frenchmen took a cooler view of the subject than their colleagues across the border in Germany. Nationalism no doubt played a role: the Franco-Prussian War and the resulting loss of Alsace-Lorraine lay only a few decades in the past, and World War I loomed on the horizon. But that role was only secondary. Henri Poincaré (1854-1912), the doyen of French mathematics and Hilbert's chief rival for the title of "greatest living mathematician," had arranged a translation of Cantor's works a few years earlier. Poincaré expressed reservations toward Cantor's "philosophical turn of mind," worrying that French readers would object to "research which is at the same time philosophical and mathematical, and where arbitrariness has an excessive place" [GK09, qtd., p. 30]. Poincaré thought the higher reaches of Cantor's transfinite realm "have a whiff of form without matter, which is repugnant to the French spirit," though he nevertheless recognized its potential mathematical importance [GK09, qtd., p. 30].

Two young mathematicians in the audience at the 1900 congress shared the attitude of cautious acceptance taken by Poincaré: the upwardly mobile Émile Borel and his younger friend and colleague René Baire [GK09, p. 36]. Each had previous exposure to Cantor's ideas. Beginning in 1898, Baire studied under Samuel Giuseppe Vito Volterra (1870-1940) and Giuseppe Peano (1838-1952) in Italy. Both Italian mathematicians had read Cantor's works in the German, and Peano had begun investigations into set theory with the aid of his formalization of mathematical language [Pea67, p. 83]. Borel had made use of Cantor's set theory in his thesis, published in 1894 [GK09, p. 41]. Their early interest in the subject, as well as their sentiments toward its legitimacy, were shared by Lebesgue, who was then just starting as a professor in Nancy and probably too poor to attend the congress [GK09, p. 35]. In a series of four works, this trio of French mathematicians would bring into being the subject of descriptive set theory,[2] at the same time cementing set theory's role as a fundamental tool for analysis.

3.1.1 Borel: *On a Few Points Regarding the Theory of Functions* and *Lessons on the Theory of Functions*

Borel was an indefatigable academic and citizen. As a young man, he placed first in the examinations for both the prestigious École Normale Supériure and École Polytehcnique; he would later serve as the director of the former for almost a decade. After stepping down, he served as Minister of the Navy for a decade and a half, for which he was later imprisoned by the Vichy government [OR08]. So it is no surprise that his research career began with a certain amount of fanfare. Following the development of the modern conception of a function of a real variable as an arbitrary correspondence between real numbers, the construction of pathological counterexamples came into vogue. The project of constructing functions which behave in unexpected ways—e.g., being continuous but nowhere differentiable, bijecting the plane with the line, or being discountinuous at rational numbers but continuous at irrational numbers—developed as a natural result of the expanding range of the function concept, and garnered the attention of Riemann, Cantor, Peano, Karl Theodor Wilhelm Weierstrass (1815-1897), as sketched in Subsection 2.2.4

It was against this backdrop that Borel published his 1894 thesis *On a Few Points Regarding the Theory of Functions*.[3] Borel's stated aim was "to show that there are interesting functions with simple properties, apart from the analytic functions proper" [Bor94, p. 39]. For essentially any desired property, Borel noted that "[a] skilled analyst will often be able to manufacture a function that does not possess it" [Bor94, p. 39]. But by restricting their attention merely to the accumulation of such negative results, Borel claimed that his predecessors had failed to develop any positive theory of the general properties of even the simplest non-analytic functions [Bor94, p. 39].

Borel's thesis, then, is a double contribution to this project. In it, he investigated the possibility of analytically continuing certain complex functions, the closure of whose poles form simply closed curves or lines; and the construction of real functions everywhere possessing derivatives of all orders but nowhere equal to their Taylor series [Bor94, pp. 7, 27]. But the work was revolutionary also in its tools: in it, Borel made essential use of Cantor's theory of sets. Following Cantor, Borel exploited the interplay of two notions of size—cardinality and total length—to produce new proofs of the uncountability of an interval and Liouville's Theorem [Bor94, pp. 43, 44-47]. He also showed that, "if one has on a given interval an infinity of partial intervals, such that every point of the given [interval] is in the interior of at least one of the intervals, one can determine *effectively* a finite number of intervals ... having the same property" [Bor94, p. 43; emphasis is mine]. Borel demonstrated this theorem, now known as the Heine-Borel Theorem, by inductively building the

[2]It bears noting that no one, at this point, had called the subject "descriptive set theory." That would have to wait until Luzin and the Russian analysts took up the subject.

[3]*Sur quelques Points de la Théorie des Fonctions.*

subcover, starting at the left endpoint of the given interval, and at each stage choosing a new interval containing the right endpoint of the last. The crux of Borel's proof is, "assuming the intervals [to be] numbered according to some law," that the process will either terminate or produce an endpoint corresponding to every number of the second number class[4]—a set of cardinality $\aleph_1$, and hence strictly larger than the original collection of intervals [Bor94, pp. 43-4]. "Thus one will, in employing the regular process indicated, *determine effectively* a finite number of intervals that cover all of the given [interval]" [Bor94, p. 44].

This tantalizing hint of set theory's power captured his interest. The mathematically fruitful synthesis provided the blueprint for his 1898 *Lessons on the Theory of Functions.*[5] Borel's choice of title belies his purpose: observing the dated nature of existing texts on the theory of functions, he writes:

> "It has seemed to me that it was possible to try to do useful work by attempting to expose, in an elementary manner, certain investigations which, although relatively recent, take on more important importance every day. Among these is the theory of sets; it is to this that this work is consecrated. I have, however, endeavored to give it the title *Lessons on the Theory of Functions*, for, in speaking of sets, I sought never to lose sight of the applications" [Bor98, p. vi].

Indeed, the continuity with his 1894 thesis extends to the major innovation in *Lessons*: the theory of measure.

As investigated in Chapter 2, one can trace many mathematical developments leading to the 20$^{\text{th}}$ century to the problem of determining areas and lengths. However, in contrast to the development of the function concept, the question of what area *is* did not arise until very late in the 19$^{\text{th}}$ century. Domains complicated enough to challenge the pre-theoretic understanding of "area" had not, until that point, arisen in mathematical practice. Only through the study of point sets begun by Cantor[6] could one encounter genuine counterexamples that would legitimize such an investigation.

Earlier mathematicians, such as Cantor, Marie Ennemond Camille Jordan (1838-1922), and Peano had proposed various definitions of "content," motivated in part by a desire to measure how discontinuous certain functions were [Kline, 1042]. But the early attempts suffered from various theoretical shortcomings that hobbled their application. Borel was able to overcome these faults no doubt because of his particular emphasis on calculation. Measure, for Borel, consisted in an inductive procedure. One begins by noting that intervals have lengths which can be immediately calculated. One can then employ the intervals to measure sets of a slightly more complicated form:

> "When a set is formed entirely of the points contained in a countable infinity of intervals that do not encroach upon one another, and which have total length s, then we say that the set has measure s" [Bor98, p. 46].

[4] Recall that "the second number class" was Cantor's term for what today are referred to as countable ordinals.

[5] *Leçons sur la Théorie des Fonctions.*

[6] Or, perhaps, those investigators like Hankel whose work led immediately into Cantor's.

Other simple constructions yield further sets which one could measure:

> "More generally, if one has a countable infinity of sets which pairwise have no points in common and have respectively as their measures s_1, $s_2, \ldots, s_n, \ldots$ their sum (or the set formed by their union) has for its measure
>
> $$s_1 + s_2 + \cdots + s_n + \cdots$$
>
> ... and if a set E has for its measure s and containing all of the points of a set E' whose measure is s', then the set $E - E'$ ... will be said to have measure $s - s'$" [Bor98, pp. 46-47].

Measurable sets were, for Borel, simply any set whose measure one could eventually calculate using this process. He concedes that his definition is given "without our intending to imply that it is not possible to give a definition of the measure of other sets; but such a definition would be useless; it might even embarrass us" [Bor98, p. 48]. But the question of characterizing of the measurable sets in general seems not to have occurred to Borel; or, if it did, he probably thought any such characterization would be meaningless for reasons we shall explore in Section 3.2.

Borel did carry his analysis far enough to determine that one could measure any perfect set [Bor98, p. 49]. But the question of whether Borel had indeed pursued the notion of measure to its completion would have to await a later investigator.

3.1.2 Baire: *On Functions of Real Varaibles*

Unlike his older companion at the International Congress, Baire would not find spectacular professional success. Laid low by recurrent physical illness and, in his own words, a "debilitating" psychological disorder, Baire spent most of his life a series of unsatisfying teaching positions away from the capital [OR00]. Embittered by a lack of recognition—recognition which *had* been showed to the younger Lebesgue, much to Baire's dismay—Baire ultimately passed away in difficult financial straits, similar to those in which he had grown up [OR00]. His snubbing was, by all accounts, undeserved. Borel, in his thesis, initiated a program of trying to investigate the *general* properties of arbitrary functions. This is the project Baire took up in his 1899 thesis *On Functions of Real Variables*.[7]

While the systematic study of irregular functions was quite recent, Baire saw that his program of research fit within a larger mathematical narrative:

> "The word *function*, which served originally to designate the different values [*puissances*] of the same quantity, took on a better and better understood signification, up to the moment when Dirichlet gave to this word the sense which one attributes to it today. There is a function as soon as one imagines a correspondence between numbers ... One does not occupy oneself ... with investigating by what means the correspondence can be effectively established ... [And in] a general manner, a set of well-defined properties that one imposes on a function being given, it is necessary to study as completely as possible the properties which are the necessary consequences of the former" [Bai98, pp. 1-2].

[7] *Sur les Fonctions de Variables Réelles.*

Comfortably situated in the analytic tradition, Baire set himself two problems: to determine the nature of a function $f(x, y)$ of two real variables, continuous in x for each fixed y and *vice versa*, when restricted to some line $ax = by$; and to determine which functions can be characterized as the limit of a series of continuous functions [Bai98, p. 2].[8]

The resolution that Baire provides to these two questions exemplifies the utility of the set-theoretic framework in analysis. In both cases, Baire completely characterizes the functions in question: they are those which possess points of continuity relative[9] to any perfect set E [Bai98, pp. 28, 62]. Having completely solved his original problem, Baire investigates several generalizations. The first of these is to note that if one denotes by "class 0" the continuous functions, then the limit of a series of class 0 functions may be called a "class 1" function. By his main result, these are precisely the functions having a point of continuity relative to any perfect set. Can one characterize class 2 functions, the limits of series of class 1 functions? What of class n functions?

The answer, Baire observes, is not so simple: "[i]t would be interesting to demonstrate the effective existence of functions belonging to different classes ... and to characterize each of these classes ... [but] the difficulties become very great as soon as one takes on the study of class 2 functions" [Bai98, p. 71]. To begin, the construction need not end in the finite, as Baire observes, but rather can be continued into the transfinite: functions in class ω are limits of series of functions of finite classes, functions in class $\omega + 1$ are limits of series of functions of class ω, and so forth. The construction can be continued through the entirety of the second number class to generate a collection E with the property that any series constructed with functions from E converged to a function in E.[10] Even so, the difficulty of characterizing even the lowest levels of E hinted at the difficulty of studying arbitrary functions in general: the cardinality of E is $2^{\aleph_0}$, far smaller than the cardinality of *all* functions of a real variable, $2^{2^{\aleph_0}}$, and therefore "despite being better understood than the continuous functions, does not form a very special [*particulier*] category with respect to the set of all functions of which one can conceive" [Bai98, p. 71].

Baire, like his predecessors, was also considered the problem of characterizing how discontinuous a function is on the basis of the size of its set of points of discontinuity. His approach, however, was not to consider the density, cardinality, or measure of a set, but rather something entirely novel:

> "[O]ne sees that these remarks lead completely naturally to the study of a subset of the line satisfying the following properties: there exists a countable infinity of sets $P_1, P_2, \ldots, P_n, \ldots$, every one of which is nowhere dense [*non dense*], and such that all of the points of P are part of at least one of $P_1, P_2, \ldots, P_n, \ldots$. I will say that a set of this nature is of *first category*. All of those sets which do not possess this property will be said to be of *second category*" [Bai98, p. 65].[11]

[8]That is, point-wise limits considered without regard to uniform convergence.

[9]That is, the restriction of the function to any perfect set has a point of continuity in the relative topology.

[10]That is, if, for each $n \in \mathbb{N}$, $f_n \in E$, then $(\sum_{n=0}^{\infty} f_n(x)) \in E$.

[11]The essential difficulty, however, is not the mere identification of the notion of category, but rather showing that it is a *real* distinction. This is the problem he solves in proving the Baire category theorem, which demonstrates that there are sets which are first category but not second,

While Baire devotes little of his work to notion of category, it would ironically constitute his most lasting contribution to mathematics. Unlike measure, category is inherently topological and hence applicable in a wide variety of contexts: "generic," a ubiquitous term in modern mathematics, is synonymous with "having a complement of first category."

3.1.3 Lebesgue: *Integral, Length, Area* and *On the Analytically Representable Functions*

One might have imagined that Borel's notion of measure would have come to play a role similar to Baire's notion of category. It might well have, were it not for Lebesgue, the third and final luminary of traditionalist French set theory. Like Baire, Lebesgue grew up in modest circumstances [GK09, p. 45]. His talent for mathematics, however, was evident from a young age, earning him the nickname "the Aristocrat of Geometry" [GK09, p. 45]. Indeed, geometry would serve as Lebesgue's mathematical muse all his life. His achievements stemmed from exploring mathematical liminal spaces, the "connections which I feel to be very close between the general theory of functions and pure geometry, [though] they remain a little mysterious" [GK09, qtd., p. 47]. Having finished his studies under Borel at the *École Normale Superieur* in 1898, it was in this spirit that for the next two years Lebesgue Baire's recent work on discontinuous functions while working in the university's library [HL02, p. 3].

Borel and Baire had pursued a line of research premised upon the potential of set theory to remedy long-standing difficulties in analysis deriving from its usefulness as a framework for taxonomies and the mollification of pathologies. Lebesgue's insight was to see how deeply these difficulties ran. As was the case for Baire and Borel, Lebesgue's influential contributions to mathematics started with his thesis, *Integral, Length, Area*,[12] on which he was examined in 1902. In it, Lebesgue framed his task in the following way:

> "We know that there are derived functions that are not integrable when one adopts ... the definition of an integral given by Riemann; with the result that integration, as defined by Riemann, does not in all cases, let us solve the fundamental problem of integral calculus: How to find a function if we know its derivative" [Leb07, p. 1212].

This question, Lebesgue noted, is intimately related to another: attaching numbers representing length or area to point sets [Leb07, p. 1212]. Lebesgue felled both at one blow with a refinement of the "somewhat cursory indications given by Mr. Borel" and Jordan [Leb07, p. 1212].

While Borel had described a recursive process by which one measures a set, Lebesgue saw how a measure could be found, in some sense, all at once. Any set E ought to be smaller than any finite or countable set of intervals whose union contains it. One attains, therefore, an upper limit on the measure of E by letting $m_e(E)$ be the lower limit of the total lengths of countable sets of intervals covering E [Leb07, p. 1217].[13] To ensure that $m_e(E)$ is a good enough approximation to E's size—that

and *vice versa* [Bai98, p. 65].

[12] *Intégral, Longeuer, Aire.*

[13] In symbols, $m_e(E) = \inf_{\{I_n\}\in\mathcal{I}} \sum_{n=0}^{\infty} \ell(I_n)$, where $\mathcal{I}$ is the collection of all countable collections of intervals $\{I_n\}$ such that $E \subset \bigcup_n I_n$.

is, to ensure that E is not itself small but somehow "stuffing" larger coverings—one simply asks that $m_i(E) = (b - a) - (m_e([a, b] - E))$ for some interval containing E, equal $m_e(E)$. Indeed, as Lebesgue notes that "if the problem of measure is possible, the measure of a set E is contained between the two numbers $m_e(E)$ and $m_i(E)$ that we have just defined" [Leb07, p. 1217].

Lebesgue's definition is more general than Borel's in that it allows for the measurement of a greater number of sets: as Lebesgue points out, every subset of the Cantor set—of which there are $2^{\mathfrak{c}}$ [14]—is measurable, while there are only $\mathfrak{c}$-many Borel measurable sets, since each is determined by a countable number of constraints [Leb07, p. 1219]. However, Lebesgue's measure lacked a definite procedure that would allow one to calculate it, as it is "in fact defined by considering an uncountable infinity of numbers" in the form of the total lengths of coverings [Leb07, p. 1221]. Moreover, as Lebesgue observed, his measure, in some sense, merely completed what Borel had already accomplished: for any Lebesgue measurable set E, there are Borel sets[15] F and G such that $F \subset E \subset G$, and $m(F) = m(E) = m(G)$ [Leb07, p. 1220]; Lebesgue's contribution was merely to notice that one could profitably call E's measure the measure of F and G.

Even so, Lebesgue's definition offered a new perspective on the measure problem that immediately proved its worth: with it, he met his stated goal of recovering much of the Fundamental Theorem of Calculus in his thesis, obtaining the essentially complete form by 1904 using the same methods [HL02, p. 6]. Further advances made on its basis in the theory of trigonometric series and other areas established it as a revolutionary notion [HL02, p. 7]. By the time of his death, Lebesgue's thesis would have entered the canon as "one of the finest any mathematician has ever written" [HL02, qtd., p. 2].

Nevertheless, many questions remained: "I know of no function that is not summable [i.e., integrable], not being aware of whether any exist… [Furthermore,] in no sense has it been shown that the problem of measure is impossible for sets (if there are any) whose interior and exterior measures are unequal" [Leb07, pp. 1213, 1218]. The complexities of Baire's function classes and Borel's measurable sets were largely unexplored, as well, with even the most basic facts still unknown. How complicated can a Borel set be? Does the Baire hierarchy simply stop growing at some point in the second number class, or even the first?

Lebesgue would turn to these matters in a 1905 paper entitled *On the Analytically Representable Functions*.[16] The heart of this work was a remarkable connection stemming from a synthesis of the techniques of Borel, Baire, and Cantor.

In analogy with Baire's definition of function classes, Lebesgue saw that one could arrange the Borel measurable sets into a hierarchy based on the complexity of implementing Borel's measuring procedure. Lebesgue observed that Baire's function classes were formed by successive attempts to "close off" the current level under infinite series, beginning with the continuous functions. The same held true of Borel's measuring procedure, which could be broken into successive applications of

[14] The notation $\mathfrak{c}$, short for "continuum," is merely another way of writing the cardinality $2^{\aleph_0}$.

[15] "Borel set" is simply another term for "Borel-measurable set"; a third term was also in use during this period: the "B-measurable sets" (*ensembles mesurables* B).

[16] *Sur les Fonctions Représantables Analytiquement.* "Analytically representable" here means being in the Baire hierarchy. Lebesgue is thinking of the infinite series of continuous functions that Baire introduced as a generalization of more familiar analytic representations, such as a function's Taylor series or Fourier series expansion.

three kinds of steps in an analogous way:

> "The operation I gives the set formed by the points belonging to at least one of a given family of sets. The set so obtained is called the sum of the given sets... The operation II gives the set formed by the points common to all of the sets of a given family of sets. The set so obtained is also called the common part of the given sets... The operation II may ... be replaced by operation I and a new operation, operation III, permitting the passage from a set to its complement" [Leb05, p. 158].[17]

But unlike the Baire hierarchy, the stratification took on a complicated, bipartite form. The end result of applying Operations I, II, and III depended upon a choice made at the very beginning: whether one began with open intervals or closed intervals. The hierarchy therefore was divided into two towers: the O sets and the F sets. The class 0 O sets were merely the open sets, and the class 0 F sets were the closed sets. Operation III took an O set of class α to an F set of class α and vice versa; operation I, applied to all countable families of O sets of class α, yielded the F sets of class $\alpha + 1$, while operation II was used in the same way to obtain the O sets of class $\alpha + 1$. If α was a limit ordinal[18] then the O sets of class α were simply all those sets of class β for some β less than α, and similarly for the F sets. The O and F sets of class α for all α less than ω_1 were simply the Borel measurable sets.

Remarkably, the analogy between Baire's hierarchy and the newly constructed hierarchy of Borel sets went much deeper: the two were, in fact, *the same.* The pre-image of any function of any open interval by a function of Baire class α is an O set of class α; if the interval were closed, the result would be an F set of class α [Leb05, pp. 156-7]. Armed with this correspondence, Lebesgue showed that for every α less than ω_1, there were functions of class $\alpha + 1$ that were *not* of class α; consequently, the same held for Borel sets.

Using the hierarchy and a generalization of Cantor's diagonalization argument, Lebesgue in fact managed to *construct* a set of each Baire class; by carrying the argument through to ω_1, Lebesgue remedied a perceived short-falling of his thesis, constructing an explicit example of a function *not* analytically representable and hence a set *not* Borel-measurable [Leb05, p. 214].

Lebesgue's study of the analytic representations of functions was aimed at completeness. The difficulty of creating the non-analytically representable function evinced a fundamental fact: beyond operations I, II, and III, Lebesgue showed that essentially any way one might hope of producing a non-Borel measurable set could not succeed. But matters were less certain than they seemed, and what his 1905 paper failed to resolve would open the door to a whole new school of mathematics.

3.2 The Traditionalist View of Set Theory

The relationship of the traditionalists to the theory of sets that provided the grist for their mathematical discoveries was a tense and, at times, dissonant one. The dissonance is, in some regards, jarring: on the one hand, set theory was the core of their

[17]In modern notation, from $\{A_n\}$ operation I yields $\bigcup_n A_n$, and operation II yields $\bigcap_n A_n$. Operation III transforms A into $\mathbb{R} \setminus A$. All of the families of sets are assumed to be countable.

[18]Recall that a limit ordinal is one that has no direct predecessor, e.g., ω.

mathematical research program, and their work was decisive in bringing set theory into the mathematical mainstream; on the other, they felt deep mistrust toward many of set theory's central edifices—particularly Cantor's theory of cardinals and ordinals up to and including $\mathfrak{c}$ and ω_1, and later, Ernst Friedrich Ferdinand Zermelo's (1871-1953) Axiom of Choice—and largely disavowed them, setting boundaries to set theory's use that persist even in contemporary mainstream mathematical practice.

The tension is, in some regards, puzzling: in some cases, the traditonalists carried their criticisms of Cantor's even to elements of the theory that were, unbenknownst to them, indispensable for their own work. And the positions they staked were, to a certain degree, philosophically unrigorous, or, at least unsophisticated. But their cautious regard for set theory was not irrational—rather, the traditionalists were guided by the mathematical values they absorbed from their forbears. The traditionalists knew, consciously or unconsciously, that the measures of mathematical legitimacy for any new theory were applicability to established mathematical problems and the possibility of carving out a domain in which the theory could be applied without fear. In this regard, the traditionalists were reacting to two opposing forces: the proliferation of set theoretic antinomies, which lent the theory a profound *internal* instability; and set theory's untapped potential to reshape analysis. Their position was a compromise, a way of laying out a safe fragment of set theory which could be gainfully applied in a long-established and important research tradition without risk of encountering the thorny—and, importantly, unmathematical—issues that came from accepting the theory as a whole.

3.2.1 The early period: cautious acceptance

Even though the views of Baire, Borel, and Lebesgue shifted while their school of set theory was in full flourish between 1894 and 1905, the essential characteristics of their mature views were already present in their earliest publications. Three characteristics are particularly salient in this regard: a generally suspicious attitude toward set theory, a special regard for *effectiveness*—a term that for the French analysts took on a special meaning—and a profound concern for the applicability of their research. Each of these three characteristics was a manifestation of an essential belief in a realm of *real* or *legitimate* mathematics, toward which mathematical endeavors ought to aim.

Set-theoretic unease

Borel begins *Lessons* with the following goal: "to give the notion of a set the precision necessary in order to use it in rigorous research" [Bor98, p. 1]. The central question, in his view, is, when can one consider a set as *given* [Bor98, p. 2]. Cantor's answer—when one can determine *intrinsically* and on the basis of the law of excluded middle whether a given element belongs to the set or not—simply will not do [Bor98, p. 2,fn. 1]. Rather, "[w]e shall say that a set is given when, by any means, we know how to determine all the elements one after the other, without excepting one and without repeating any of them several times" [Bor98, p. 3]. Borel offers reasons for his caution:

> "[O]ne of the ideas that we should be most fortunate to give to the reader who wishes to think for himself on the theory of functions is that, in all

> questions to which the infinite appertains, one must be extremely wary of alleged clarity: nothing is more dangerous than to rest content with empty words in such matters" [Bor98, p. 3].

His trepidation was well-warranted. Since the publication of Borel's thesis, Cesare Burali-Forti's (1861-1931) modestly titled *A question on Transfinite Numbers* had caused a stir in the mathematical community. The crux of the article was the following observation: if one takes Ω to be the set of all ordinals, then Ω itself is a well-ordering, to which some ordinal α corresponds. Since $\alpha \in \Omega$, Ω must be an initial segment of one of its own initial segments, something Cantor had independently proven impossible in the same year [Can52, p. 144]. The failing was, in Burali-Forti's opinion, substantial: "It seems that the order types thus fall short of one of their most important objectives" [BF67, p. 111].

Burali-Forti's discovery was a shattering blow to a theory that had raised hackles from the outset; it was made only the more terrible by the rapid discovery of further paradoxes by Bertrand Arthur William Russell (1872-1970), Jules Antoine Richard (1862-1956), and Julius König (1849-1913). The "intrasubjective immanent reality" that Cantor had thought secured his theory seemed to grow feebler with each passing day [Can05a, §8.1].

Thus, it is no surprise that Baire too, fearful of building on sand, dispenses with any set theory he can do without:

> "I will point out once and for all on this subject that we shall never have to worry about the difficulties included in the abstract notion of *transfinite number*... In actuality, for example, the set P^α, α being a determined number of the second number class, represents something perfectly determined independent of all abstract considerations relating to the symbols of Mr. Cantor" [Bai98, p. 36].

While he and Lebesgue display a greater comfort than Borel with the set theoretic tools they adopt, none share in the exuberance for set theory displayed by Cantor and his other early followers.

Effectiveness

Effectiveness was a key mantra for the French analysts. For instance, Borel writes that his proof of the Heine-Borel Theorem, while more complicated than other proofs, has the chief virtue that it demonstrates that "one can choose *effectively* a limited number of intervals" covering the given interval [Bor94, p. 43; emphasis mine]. In *Lessons*, he is concerned to give an "effective" demonstration of the existence of sets having uncountable cardinality, and praises Charles Hermite's (1822-1901) demonstration of the transcendence of e: "It was, in effect, the first *effective* example, if one may so speak of it, of a transcendental number; that is to say, the first example of a transcendental number defined in a simple way by analysis and not only by arithmetical series" [Bor98, p. 25].[19]

[19] Borel is contrasting Hermite not only with Liouville's construction, which he found unnatural, but also the set theoretic proofs, which, with regard to effectiveness, were even more objectionable. Of course, Borel invented one of the set theoretic proofs, in which one shows that the algebraic numbers have measure zero. This is a typical example of the kind of dissonance with which Borel in particular contended.

Similarly, Baire extols the interest of "effectively demonstrating the existence of functions belonging to different classes" [Bai98, p. 71]. Lebesgue, for his part, draws much of the justification for the project of his 1905 *On the Analytic Representation of Functions* from Baire's failure to do just that. The construction of explicit, concrete examples was to be preferred to the cardinality-based arguments he had adduced to show the existence of a Lebesgue measurable set that was not Borel, and that Baire had adduced to show that there was a function not in any function class.

What, exactly, was meant in this period by "effectiveness"? Borel's *Lessons* provides some indication:

> "It seems to us that this is an axiom which must be admitted in the same way as the axiom of Archimedes, and in a very general manner; It is in any case certain that this proposition is not dubious, by the words, 'any function,' one understands a function which can be effectively defined, that is to say such that one can, by a limited number of operations, calculate, with a given approximation, its value for a given value of the variable" [Bor98, p. 117].

In parallel, Baire remarks that the advantage of his system of classification of discontinuous functions is that they grow "more and more complicated, but [are] always capable of being tethered in a very precise manner to the continuous functions" [Bai98, p. 70]. Indeed, it is this connection—borne out in a finite number of steps by the fact that, since no decreasing sequence of ordinals can be of infinite length—that Lebesgue used to such effect in *On Analytically Representable Functions*.

Effectiveness, in short, meant the potential to actually be carried out explicitly and concretely. The standard was not a metaphysical one, but rather heuristic, and determined in reference to past mathematical practice: Hermite's proof of the transcendence of e was effective because it involved something seemingly more specific and natural, i.e., the limit of the sum

$$\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \tag{3.1}$$

than the more anonymous Liouville numbers; likewise, arguments based upon arbitrary functions and cardinalities lacked the meat of demonstrations based upon what was more easily recognizable as a calculation. Of course, effectiveness was, for different individuals, a virtue in varying degrees: for Borel, it was law—arbitrary discontinuous functions were totally inadmissible, as one had no procedure for calculating when two were the same [Bor98, p. 125]. For Lebesgue, it was merely a signpost, pointing the way toward richer mathematics. Indeed, Lebesgue criticizes the stringency of the requirements placed by Borel upon functions, noting that they are too restrictive even to admit the second level of the Baire hierarchy:

> "In general, a calculation is illusory if it is assumed that two passages are performed successively at the limit, unless the second is related to a uniformly convergent sequence. Now, it is such a calculation that one would have to make to calculate $\chi(C)$,[20] C being given later by its decimal digits, that is to say by a series" [Leb05, p. 206].

Nevertheless, the value was a shared one, and played a signal role in shaping their mathematical endeavors.

[20]Here, χ denotes the indicator function of the rational numbers.

Applicability

The final value of the French analysts, both explicit and tacit in their work, is applicability. A piece of mathematics rose to the level of a *contribution* when it shed light on well-established mathematical problems. Thus, we find Borel, in his thesis, self-consciously attempting to "see what importance [his results on discontinuous functions] might have for applications, notably in mathematical physics" on the basis of the fact that "no demonstration has ever been given that one can apply Taylor's formula to the functions one encounters in physics" [Bor94, p. 3]. For Baire, set theory was an indispensable tool for the study of functions:

> "[A]ny problem relative to the theory of functions leads to certain questions relative to the theory of sets; and it is to the extent that these latter questions are advanced or can be advanced that it is possible to resolve more or less completely the given problem" [Bai98, p. 121].

Lebesgue, perhaps more so than any of the others, sees with acuity the importance of set theory to matters of fundamental importance in mainstream mathematics. His principal object in his thesis is "to give definitions as general and precise as possible of some of the numbers one looks at in Calculus: a definite integral, the length of a curve, and the area of a surface" [Leb07, p. 1212]. Indeed, of the three, his contribution to mainstream mathematics is perhaps the most profound, having utterly reshaped the field of real analysis [HL02, p. 3]. But it was Borel who summed up the attitude best when he wrote some years later, reflecting on set theory's increasing formality and distance from ordinary mathematical concerns, that

> "From the day set theory stops being metaphysical and becomes practical, the new ideas may produce a flowering of beautiful results… Maybe from this profusion of formal logic, which appears as a construction without any basis, one day some useful idea will come" [GK09, qtd., p. 63].

The ordinals and cardinals

Nowhere are these values more striking than in the French analysts' treatment of transfinite arithmetic. For Cantor, the ordinals were isolated from "*transsubjective* or *transient* reality" of which one's ideas of numbers form a part by virtue of occupying a completely determinate place in one's understanding [Can05a, §8.1]. For the French analysts, this explanation of what the ordinal numbers are must have seemed like empty speculation with no place in mathematics. It is no surprise then that they were consequently loathe to use the ordinal numbers: even ten years after the publication of his thesis, Lebesgue felt the need to apologize for his use of the ordinals in the second edition of *Lessons on Integration and the Search for Primitive Functions*[21] [HL02, p. 5].

Far more strongly opposed was Borel, who, in the appendix to *Lessons*, criticizes Cantor's transfinite numbers for being subject to antinomies and constructed in a circular manner. Cantor had posited three principles of generation for ordinal numbers. The second of these was that "if any defined sequence of integers is put forward of which no greatest exists … a new number is created, which is thought

[21] *Leçons sur l'Integration et la Recherche des Fonctions Primitive.*

of as the limit of those numbers" [Can05a, §11.2]. It was on this basis that Cantor deduced the existence of ω_1. Both this and the construction of the second number class as a *completed* entity concern Borel. He writes,

> "On the other hand, as [the second principle's] application [to a countable sequence in the second number class] leads only to a countable set, to which it is *again* applicable, there is an antinomy which ... cannot be resolved except by attributing a sense to the word *transfinitely* and admitting, consequently, that in applying the theorem *transfinitely* on will have a set ... of cardinality greater than the first" [Bor98, p. 121].

Of course, Borel thinks to do so is totally illegitimate: the only meaning one can assign, *a priori*, to the word "indefinitely" is "as often as there are whole numbers" [Bor98, p. 122]. One cannot assign it the meaning "as often as there are numbers of the second number class," as the second number class is assumed to be the result of "indefinitely" constructing larger numbers of the second number class [Bor98, p. 122].

For this reason, Borel denies the existence of $\aleph_1$—and is presumably why he did not attempt, as Baire, to stratify his measurable sets. He feels secure in this renunciation, because of a kind of working mathematician's continuum hypothesis: the cardinality of the continuum and the cardinality of the natural numbers "suffice for the applications we have in view" [Bor98, p. 20].

But some ordinal numbers are indispensable for his analytic task. Borel's solution is to construct all of the ordinal numbers one might need in analysis in terms of concrete, familiar mathematical objects: functions. Given a function $\varphi(n)$ growing faster than n,[22] Borel notes that

$$\lim_{n\to\infty} \frac{\varphi(n)}{n} = \lim_{n\to\infty} \frac{\varphi(\varphi(n))}{\varphi(n)} = \ldots = \lim_{n\to\infty} \frac{\varphi_m(n)}{\varphi_{m-1}(n)} = \ldots = \infty \tag{3.2}$$

and

$$\lim_{n\to\infty} \frac{n}{\varphi(n)} = \lim_{n\to\infty} \frac{\varphi(n)}{\varphi(\varphi(n))} = \ldots = \lim_{n\to\infty} \frac{\varphi_{m-1}(n)}{\varphi_n(n)} = \ldots = 0 \tag{3.3}$$

where $\varphi_{m+1}(n) = \varphi(\varphi_m(n))$. Thus, the functions n, $\varphi(n)$, $\varphi_1(n)$, $\ldots$, $\varphi_m(n)$, $\ldots$ can be ordered by their rate of growth in exactly the same manner as the natural numbers. Now, the function $\psi(n) = \varphi_n(n)$ has the property that

$$\lim_{n\to\infty} \frac{\psi(n)}{\varphi_m(n)} = \infty \qquad \lim_{n\to\infty} \frac{\varphi_m(n)}{\psi(n)} = 0 \tag{3.4}$$

Thus $\psi(n)$ stands in relation to the ordering n, $\varphi(n)$, $\varphi_1(n)$, $\ldots$, $\varphi_m(n)$, $\ldots$ exactly as ω, in Cantor's system, stands in relation to 1, 2, $\ldots$, m, $\ldots$ The function $\varphi(\psi(n))$ can go proxy for $\omega + 1$, and similarly, one can build a countable collection of functions having the order type of any countable ordinal. Moreover, this construction—effected by means of a theorem of Paul David Gustav du Bois-Reymond (1831-1889)—as opposed to Cantor's second principle of generation, "is not a postulate; it is a *mathematical fact* that does not rest upon any *a priori* consideration" [Bor98, p. 121]. It has the additional advantage that "its power is much more limited [than

[22] Borel actually construes the functions as functions of a real variable; this presentation is merely simpler.

Cantor's principle]; it carries within itself its bounds [*principe d'arrêt*], for it is not applicable except as far as the set already obtained is countable" [Bor98, p. 121].

Lebesgue employs the same gambit in *On the Analytically Representable Functions*, writing,

> "I wish to say why the application, used in this classification, of the transfinite numbers does not raise, in my opinion, any difficulty. If one studies the growth of functions and if, having characterised the growth of x^n by n, one notes that e^x grows faster than x^n, one could feel the desire to characterize this new growth rate by a new symbol, ω. No one will see any inconvenience... Besides, the classification of Mr. Baire... can, as the theory of the growth of functions, provide a solid base for the theory of transfinite numbers" [Leb05, pp. 142-3].[23]

It is in this manner that the French analysts rid themselves—or attempt to rid themselves—of anything appertaining to Cantor's infinite not absolutely necessary for the study of analysis.

3.2.2 The late period: Poincaré and Zermelo

Poincaré and real mathematics

Borel, Baire, and Lebesgue were influenced a great deal by Poincaré. Already in 1894, Poincaré had begun to lay out his philosophical views on mathematics, praising intuition—intended in a Kantian sense—as the foundation of mathematics, and the source of mathematical meaning and progress. His opponents were the Logicists, a group of mathematicians who sought to remake mathematics purely on the basis of logic. To Poincaré, this was totally unacceptable: "Pure logic could never lead us to anything but tautologies; it could create nothing new; not from it alone can any science issue" [Poi05a, §3].

Poincaré's opposition was informed, in large part, by a perceived disconnect between the practice of logic and the practice of mathematics:

> "If you are present at a game of chess, it will not suffice ... to know the rules for moving the pieces. That will only enable you to recognize that each move has been made conformably to these rules, and this knowledge will truly have very little value. Yet this is what the reader of a book of mathematics would do if he were a logician only" [Poi05a, §8].

The rapidly developing methods of formal logic, and its accompanying suite of esoteric symbols, seemed, to Poincaré, to miss the essence of mathematics.

Poincaré's view was a deeply psychologistic one [Gol88, p. 63]. Poincaré construed mathematics as constructive mental activity, undergirded by those truths of which one is immediately convinced [Gol88, p. 63]. In this, he did not differ from, for instance, Cantor or Dedekind, but he bitterly opposed them. The antinomies

[23] There is actually something slightly more problematic about what Lebesgue is proposing than the quotation above indicates. Recall that it is in *On the Analytically Representable Functions* that Lebesgue proves that the Baire classes corresponding to different ordinals are actually different. To do so, he actually requires the existence of the ordinals for which he is here proposing the Baire classes could replace.

of set theory, Poincaré proposed, were merely a byproduct of mathematics which had become too disconnected from the realities of mathematical practice: while the logicians struggled to resolve the paradoxes, Poincaré maintained that “True mathematics, where one does not wallow in the actual infinite, is not in question” [Poi05b, §13]. Nonetheless, Poincaré recognized the importance of set theory; like Borel, he merely thought that the Cantorians had erred in forgetting that “There is no actual (given complete) infinity... It is true that Cantorism has been of service, but this was when applied to a real problem whose terms were precisely defined, and when we could advance without fear” [Poi05b, §15].

Zermelo’s principle and the mature formulation of the traditionalist view

Poincaré’s objections, though they began in the 1890’s, crescendoed only after the turn of the century. They reached their loudest in 1905, following the publication in 1904 of *Proof that every Set can be Well-Ordered*, a short paper by Ernst Friedrich Ferdinand Zermelo (1871-1953). In it, Zermelo provides a proof of what Cantor considered a “fundamental law of thought of great consequence” [Zer67c, qtd., p. 139]: that every set can be brought into the form of a well-ordered set. Zermelo’s proof depends upon what would become known as *Zermelo’s Principle* or *the Axiom of Choice*: given any collection of non-empty sets, one can simultaneously choose from each a single element. This seemingly innocuous, even self-evident, statement soon caused a stir, drawing the criticism of such eminent mathematicians as Peano, Borel, and Poincaré [Poi05b, §13], [Zer67b, p. 186].

Zermelo’s goals were quite different from those of the French analysts. “Set theory,” he wrote, “is that branch of mathematics whose task it is to investage mathematically the notions ‘number’, ‘order’, and ‘function’ ... and to develop thereby the logical foundations of arithmetic and analysis” [Zer67a, p. 200]. The antinomies had threatened the theory, but by grounding the sets in explicit axiomatics, one could “retain all that is valuable in ... the entire theory created by Cantor and Dedekind” [Zer67a, p. 200].

But the traditionalists, who viewed set theory as playing quite a different role in mathematics, did not agree, and rejected his axiom of choice. Their reasons for doing so evinced Poincaré’s influence. Borel, one of Zermelo’s quickest critics, saw the situation in characteristic fashion:

> “It seems to me that the objections that one can raise against [Zermelo’s proof] apply equally well against any reasoning where one supposes an *arbitrary choice* to be made a non-denumerable infinity of times; such reasonings are outside the realm of mathematics” [BBLH05, p. 1077].

But he was not without his detractors. Jacques Salomon Hadamard (1865-1963) wrote to Borel soon after, posing the following objection:

> “What is certain is that Zermelo provides no method to carry out *effectively* the operation which he mentions, and it remains doubtful that anyone will be able to supply such a method in the future. But the question posed in this way (the effective determination of the desired correspondence) is none the less completely distinct from the one that we are examining (does such a correspondence exist?). Between them lies all the difference, and it is fundamental, separating what Tannery

> calls a *correspondence* that can be *defined* from a correspondence that can be *described*" [BBLH05, p. 1078].

Foreseeing the potential importance of their conversation, Borel forwarded the letter to Baire and Lebesgue. What followed was an profoundly revealing exchange of letters, in which each of the French trio revealed the distillation his ideas had undergone in a few short years. While Borel still did not believe in the uncountable, Baire, who had proudly advanced knowledge of "arbitrary functions" only a few years before now went further than Borel, maintaining that all infinities, even countable ones, are merely "in the realm of *potentiality*... [D]espite appearances, in the last analysis everything must be reduced to the finite" [BBLH05, p. 1080]. Lebesgue, for his part, intoned that "I do not grant ... any validity to the argument showing that a set which is not finite has a denumerable subset. Although I seriously doubt that a set will ever be named which is neither finite nor infinite, it has never been proven to my satisfaction that such a set is impossible" [BBLH05, p. 1083].

In their discussions, one question emerges as fundamental: what is required to prove existence in mathematics? All three object on the basis that, in the sense they find meaningful, one may not even succeed in choosing a *single* element from an arbitrary set. In Lebesgue's words,

> "I use the word *to choose* in the sense of *to name* ... So as to convey more clearly the difficulty that I see, I remind you that in my thesis I proved the existence ... of sets that were measurable but were not Borel-measurable. Nevertheless, I continued to doubt that any such set could ever be named. Under these conditions, would I have the right to base an argument on this hypothesis ... even though I doubt that anyone could ever name one? Thus I already see a difficulty with the assertion that 'in a determinate M' I can choose a determinate m'' " [BBLH05, p. 1082].

The crux of the matter is that without a *procedure*—what they might see as a determinate mental procedure—for distinguishing a single element in a set from all others, one cannot concretely effect such a choice.

Of course, Lebesgue and Baire, at least, are sensitive to the apparent contradiction that Hadamard points out between their current positions with their earlier works. Borel, for instance, implicitly called on the axiom of choice in his 1894 proof of the Heine-Borel theorem. But they rebuke Hadamard's criticism that, in focusing so pointedly on definitions, "existence is a fact like any other" independent of the way it was proven [BBLH05, p. 1084]. Borel, in an elegant summary of the traditionalist ethos, closes the conversation with the following words:

> "I prefer not to write alephs. Nevertheless, I willingly state arguments equivalent to those which you mention, without many illusions about their intrinsic value, *but intending them to suggest other more serious arguments*... One may wonder what is the real value of these arguments that I do not regard as absolutely valid but that still lead ultimately to effective results ... They have a value analogous to certain theories in mathematical physics, through which we do not claim to express reality, but rather to have a guide that aids us, by analogy, in predicting new phenomena which must be verified" [BBLH05, p. 1086].

3.2.3 The end of the French school

Borel, Baire, and Lebesgue's seminal works in analysis between 1894 and 1905 recast analysis in the image of set theory; no longer was it possible to ignore set theory as mere philosophical speculation, without mathematical substance. But just as quickly as the new techniques were ushered in, interest in them—at least in France—dried up. By 1904, Lebesgue had produced, in more or less consummate form, the solution to the problem of the Fundamental Theorem of Calculus promised in his thesis, as well as invaluable tools for analysis, such as the Dominated Convergence Theorem [HL02, p. 6]. Fubini's Theorem as well as early applications in potential theory and Fourier analysis vindicated the new conception of integral and measure: Plancherel proved his eponymous theorem in 1910 using Lebesgue's integral, providing, in some sense, a complete solution to the century old problem [HL02, p. 7], [HS05]. Lebesgue's interest in set theory began to wane around 1910, in part due to a fear that his contributions had been too systematic: "reduced to general theory mathematics would be a beautiful form without content. It would die quickly, as many branches of our science have died just at the time when general results seemed to guarantee them a new activity" [HL02, qtd., p. 10].

Borel, for his part, had begun to turn more and more toward applied branches of mathematics—especially probability and game theory—and politics [HL02, p. 9]. Though closely connected intellectually, Lebesgue and Borel's relationship had always suffered from a certain strain. Borel, whose grandfather had been a rich wool merchant and whose wife was the daughter of the celebrated French mathematician Paul Emile Appell (1855-1930)—fueling the facetious observation, common at the time, that "genius is transmitted to sons-in-law" [GK09, p. 42]—belonged to a milieu in which the more modest Lebesgue was not always at home [GK09, pp. 63-4]. Lebesgue was a Dreyfusard; Borel, a firm supporter of the military [HL02, p. 9]. Their friendship took a blow in 1912 when Borel published a paper trivializing Lebesgue's work and worsened considerably when Lebesgue was placed under his command in World War I [HL02, p. 8], [GK09, p. 64]. By 1917, their friendship had evaporated, destroyed by a bitter priority dispute, and for many years neither had returned to the set theory of his youth [GK09, p. 64]. Baire, for his part, languished in provincial teaching posts, unable to research because of a mental illness aggravated by his lack of recognition [OR00].

Before long, no new set theory was issuing from its three former stars. Borel, describing the evolution of his attitude toward set theory spoke in some sense for all three when he wrote: "Like many young mathematicians, I had been immediately captivated by the Cantorian theory; I don't regret it in the least, for that is one mental exercise that truly opens up the mind" [GK09, qtd., p. 40]. Nevertheless, important questions remained. An eager group of investigators would soon discover that Lebesgue's *On Analytically Representably Functions* had resolved less than it claimed. Indeed, even its principal achievement—the effective construction of a non-Borel measurable set—was subject to doubt on the basis of its dependance upon the existence of $\aleph_1$. With a whole world of sets now intermediate between the extremes of perfect and arbitrary sets, the possibility of revitalizing the program of research that had led to the Cantor-Bendixson theorem emerged. A new group of mathematicians who shared in many of the French analysts values—especially the priority of analysis and the importance of effective constructions—would vigorously pursue these matters.

3.3 The Russian School: 1905-1925

3.3.1 Analysis comes to Moscow

As these questions faded from the limelight in France, they would find a new home in Moscow. The link between these two centers of research was Dimitri Fedorovich Egorov (1869-1931), an instrumental player in the rise of Russian mathematics, whose efforts would help transform Moscow from an intellectual satellite to a hive of mathematical activity "contain[ing] more great mathematicians than any other city in the world" [GK09, p. 103]. Egorov's earliest contact with mathematics no doubt came from his father, the director of the Moscow Teachers' Institute, where he taught geometry [OR12], [GK09, p. 71]. Egorov received no formal schooling until he entered gymnasium, where he received a gold medal for his studies in mathematics and physics. Following graduation, he entered Moscow University in 1887, where he would remain for much of the rest of his life [GK09, p. 71]. It was there that his teacher, Nikolai Bugaev (1837-1903), introduced him to discontinuous functions, a subject which for Bugaev had been a topic of mathematical interest as often as philosophical interest [GK09, p. 68].

Egorov, however, turned to differential geometry for his early research, winning acclaim for his 1901 doctoral thesis *On a Class of Orthogonal Systems* [OR12]. Already making something a name for himself in Europe, Egorov set off to study in Berlin and Paris in 1902 [GK09, p. 71]. It was during this time that Egorov—sitting in on lectures by Ferdinand Georg Frobenius (1849-1917), Poincaré, Lebesgue, and Hilbert—began to study with great interest Lebesgue's research in the theory of functions [GK09, p. 71]. Over the following decades, Egorov would make important contributions in this area, such as what is now known as Egorov's theorem, first published as a note in *Comptes Rendus* in 1911 [Ego11]. Egorov was, moreover, a devoted teacher, and his most important contributions to the subject and to the development of Russian mathematics would come from his students [OR12].

His first and no doubt most successful was Nikolai Luzin. Born in Tomsk in central Russia, unlike many of the other mathematicians who feature prominently in this story, Luzin showed no early aptitude for mathematics, and, at least initially, performed poorly in school [OR99], [Kuz74, p. 195]. Thanks to an enthusiastic tutor from the Tomsk Polytechnical Institute, Luzin discovered a talent for creative problem-solving, and soon became the gymnasium's top student [OR99]. In 1901, he entered the faculty of physics and mathematics in Moscow, studying first under Bugaev and then Egorov. Echoing his gymnasium days, Luzin initially showed little flair for mathematics at the university. But Egorov nevertheless recognized his student's potential, and began to engage him privately in research [OR99].

Despite this, as his graduation drew near in 1905, Luzin experienced a crisis. The carnage wrought by the failed Revolution of 1905 had exacted a toll on him. In a letter to his close friend Pavel Alexandrovich Florensky (1882-1937), an able mathematician in his own right who turned instead to the cloth after graduation, Luzin wrote:

> "You found me a mere child at the University, knowing nothing. I don't know how it happened, but I cannot be satisfied any more with analytic functions and Taylor series.... To see the misery of people, to see the torment of life, to wend my way home from a mathematical meeting...

> where, shivering in the cold, some women stand waiting in vain for dinner purchased with horror—this is an unbearable sight.... After that I could not study only mathematics" [OR99, qtd.]

Seeing his star student's distress, Egorov arranged for Luzin to study abroad in France with Borel, Lebesgue, and Hadamard, much as he had—even going so far as to arrange for Luzin to stay in the same hotel in Paris [GK09, p. 81].

Ultimately, Luzin did not undergo a mathematical conversion, but rather a religious one. Luzin remained depressed in Paris, twice almost committing suicide [OR99]. But upon reading *On Religious Truth*, Florensky's thesis, Luzin remarked to Florensky that "I felt as if I leaned on a pillar... I owe my interest in life to you" [GK09, qtd., p. 83].

Supported by his new-found faith, Luzin gradually regained his interest in mathematics, obtaining an appointment as an assistant lecturer at Moscow University in 1910. At this point, Luzin shared two things with his friend and mentor Egorov: an appointment in the mathematics faculty at Moscow University, and a profound, but private, sense of faith [GK09, p. 75]. Together they would forge one of the most celebrated institutions in mathematical history.

3.3.2 The Luzitania seminar

Starting in 1914, Luzin began lecturing on optional topics outside of the standard mathematics curriculum while he worked on his doctoral thesis on trigonometric series. His informal and engaging lecture style—a stark contrast with his colleagues who read their lectures "hardly acknowledging the presence of an audience" [GK09, p. 106]—quickly attracted a circle of interested students. Meanwhile, Luzin, after his years of struggle and doubt, was coming to articulate his own interests more and more clearly. Like Baire, Borel, and Lebesgue, Luzin saw himself as an analyst. His thesis, which he presented in 1915, was entitled *The Integral and Trigonometric Series*,[24] which emphasized the historical connections between trigonometric series and the development of the notions of function and integral [Kuz74, p. 198]. His early papers were likewise devoted to the study of general problems in analytic representation by means of trigonometric and power series [Kuz74, p. 198]. Luzin, showing the influence of the French trio, framed his mathematical work thus:

> "Have the results of the theory of functions an important significance for other disciplines, and above all for classical analysis? We must bear in mind that in the present state of knowledge the method of classical analysis, the method of using analytic expressions, forms the basis of almost every mathematical discipline, and so any theory that has no direct or indirect contact with analytic expressions inevitably occupies an isolated position among the other branches of mathematics. Therefore, if we do not want the theory of functions of a real variable to be a closed theory, having no influence on other mathematical theories, we must connect analytic expressions on the one hand with the definitions and ideas of the theory of functions on the other hand" [Kuz74, qtd., p. 198].

Much like his mathematical forebears, his studies of functions were leading him ineluctably toward the theory of the sets of points on which those functions took

[24] *Integral i trigonometricheskii ryad.*

values. In 1917, he wrote: "The aim of set theory is a question of great importance: can we regard a line atomistically as a set of points? [I]ncidentally, this question is not new, but goes back to the Greeks" [Kuz74, p. 200]. Within a few years, the group of students coalescing around him would form into the celebrated Luzitania seminar, and Luzin's question would point the way for their investigations.

While remarkable in many respects, the Luzitania seminar was quite unusual on account of the diversity of its participants. Students as young as fifteen, mostly men but including a handful of women, would eagerly gather to hear Luzin and Egorov lecture in unheated classrooms, even in the depths of the Russian winter [GK09, p. 109]. Egorov, as the senior scholar, assumed a more formal air; Luzin called him "the chief of our society... [T]he definitive appreciation of our work, of our discoveries, belongs to Egorov" [Lav74, p. 176]. Luzin himself adopted a friendlier demeanor with his students, playing practical jokes and inviting them frequently to his apartment to discuss mathematics [GK09, p. 107], [Lav74, p. 176]. His approach to teaching proved an effective one. Many of the most important figures in 20$^{\text{th}}$ century Russian mathematics took part in the Lusitania seminar: Andrey Nikolaevich Kolmogorov (1903-1987), Pavel Samuilovich Urysohn (1898-1924), and Pavel Sergeevich Aleksandrov (1896-1982), among others.

3.3.3 Building on Baire, Borel, and Lebesgue

Luzin sets and descriptive set theory

From the beginning, Luzin—and consequently his students—took their cue from the works of the celebrated French analysts, pushing the geometric analysis of sets the French had initiated as far as they could. One of Luzin's first papers in this area, *On a Question of Mr. Baire*,[25] addresses a matter Baire had left unresolved in his thesis. Baire had shown that every function of a determinate class in his hierarchy had a point of continuity relative to any perfect set, after removing at most a set of first category relative to the perfect set [Luz14, p. 1258]. Luzin observed that under the assumption that $\mathfrak{c} = \aleph_1$, the converse was false. The key to his proof was the construction of an uncountable set intersecting any nowhere dense perfect set in at most countably many points [Luz14, p. 1259]. In intertwining the well-ordering of the real line[26]—of which every initial segment was countable by his hypothesis—with the closure of nowhere dense sets under countable unions, Luzin's construction became a paradigmatic use of the continuum hypothesis [Kan12, p. 418].

Problems of this genre were—in large part due to Luzin's personal interest—quickly coalescing into a self-identified discipline that Luzin named "descriptive set theory" [Kel74, p. 1]. What, exactly, spurred the development of this branch of mathematics is, however, a matter of debate. Graham and Kantor, for instance, argue that it was the Russian analysts' "own philosophical and religious traditions [specifically, the Name Worshipping sect of Eastern Orthodox Christianity to which Luzin and Egorov belonged] that pushed forward toward the creation of descriptive set theory" [GK09, p. 189]. However, I argue that an entirely different explanation is more likely. The decisive factor was rather the traditionalist ideals Luzin had

[25] *Sur un Question de M. Baire.*

[26] The critical point that enables Luzin's construction is that every initial segment is *countable* if one assumes the continuum hypothesis.

absorbed during his studies in France, coupled with the knowledge that the French analysts had sometimes failed to meet those ideals in practice.

For instance, one of the most salient ways the French analysts had fallen short was their sometimes unconscious employment of the axiom of choice in various set-theoretic constructions;[27] the particular object of Luzin's study therefore became what he termed the "effective sets," i.e., those constructed without the axiom of choice [Kel74, p. 1]. Moreover, Luzin made heavy weather of the fact that, for instance, Lebesgue's arguments were not as effective as one might desire. Writing in a 1925 note to *Comptes Rendus* entitled *On a Problem of Mr. Émile Borel and the Projective Sets of Mr. Henri Lebesgue*,[28] points to the following question posed by Borel as one of his chief motivations:

> "To give an idea of my point of view, a problem which seems to me to be most important in the arithmetical theory of continuum is: is it possible or not to define a set E such that one cannot name a single individual element of this set, that is to say, to distinguish it unambiguously from all the other elements of E" [Luz25c, p. 1318]?

While on first blush one might think that the non-Borel measurable set constructed by Lebesgue in 1905 would furnish an example, his methods were inseperable from the ordinal ω_1, and hence did not rise to the level of a true definition, at least according to traditionalist strictures. And, as we saw in Section 3.2, the existence of a such a set would have important consequences for the traditionalist mathematical worldview. The quest to solve Borel's problem would form the basis of Luzin's research program.

Luzin did not take on this project alone, however: participants in the Luzitania seminar were not only encouraged to study, but also to research. As part and parcel of the seminar's egalitarian atmosphere, Luzin frequently tasked students with solving outstanding problems in his nascent field of descriptive set theory. Luzin firmly beleived that the synthesis of set theory and analysis could be profitably carried further than even Lebesgue had imagined, and that he and his students were the ones for the task.

Analytic sets

One of Luzitania's earliest successes in this direction came in 1915 from Aleksandrov, then only a second year student at Moscow University. With the exception of a false *disproof* given by Julius König at the 1904 International Congress of Mathematicians, little progress had been made on the continuum hypothesis since the proof of the Cantor-Bendixson theorem [Kan12, p. 405]. Indeed, progress could hardly be *conceived*: until Lebesgue's isolation of the Borel hierarchy in 1905, it was not clear to what further sets one might attempt to apply a similar analysis.

But following the isolation of the Borel hierarchy, the question opened, and, at Luzin's urging, Aleksandrov applied himself to the question of whether the Borel sets satisfy the continuum hypothesis. While Lebesgue had achieved much, little was in fact yet understood about the Borel sets, as evinced by the great difficulty

[27]See, for instance, Borel's proof of the Heine-Borel theorem—which is in fact *equivalent* to a weak form of the axiom of choice—in [Bor94, p. 43].

[28]*Sur une problème de M. Émile Borel et les Ensembles Projectifs de M. Henri Lebesgue.*

of constructing non-Borel sets. However, the basic tools were in place, and in 1916, Aleksandrov published a note entitled *On the cardinalities of* B-*Measurable Sets*[29] in *Comptes Rendus*. Aleksandrov noted that for any uncountable F set E of class α,

> "we can develop E in a double-entried table
>
> $$(E) \quad \begin{cases} E_1^1 & + & E_1^2 & + & E_1^3 & + & \cdots & + & E_1^{q_1} & + & \cdots \\ E_2^1 & + & E_2^2 & + & E_2^3 & + & \cdots & + & E_2^{q_1} & + & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ E_{p_1}^1 & + & E_{p_1}^2 & + & E_{p_1}^3 & + & \cdots & + & E_{p_1}^{q_1} & + & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{cases}$$
>
> where the given set E is the common part of the set-sums situated on the horizontal lines of table (E). It is important to remark that the class of any set $E_{p_1}^{q_1}$, let it be $\alpha_{p_1}^{q_1}$, is less than α... If $E_{p_1}^{q_1}$ is not a closed set, we can develop it in an analogous table" [Ale16, p. 323].

This system of tables-within-tables then allows one to construct, corresponding to each real number between zero and one a *different* sequence of closed sets, whose intersection contains a point of E; moreover, the construction is carried out in such a way that the points corresponding to the binary sequences,[30] taken as a whole, form a perfect set [Ale16, p. 325]. Thus, the continuum hypothesis was true for all of the Borel sets for *precisely* the same reason it held of the simpler closed sets: either it was countable or it contained a perfect set. This property would come to be known as the "perfect set property," and alongside Lebesgue measure and the property of Baire,[31] would serve as the basis of much of the Russian school's future set-theoretic research.

Aleksandrov's theorem critically made use of a certain operation for knitting these Borel sets back together from their presentations as tables-within-tables. Luzin consequently posed to Aleksandrov as his next question: is every set constructed in this way a Borel set? It seemed certain that the answer ought to be yes, but the problem proved immensely difficult. Aleksandrov laboured in search of a solution for months. His lack of success sowed seeds of enmity toward his teacher that would grow throughout his career [GK09, p. 118].

But the reason behind Alexandrov's vain toil awaited discovery by a different founding member of Luzitania: Mikhail Yakovlevich Suslin (1894-1919). In 1915, Suslin was in his second year. Nevertheless Luzin suggested that he read Lebesgue's 1905 *On Analytically Representable Functions* [OR11]. After a few months' struggle, Suslin one day burst into Luzin's office with a totally unforseen discovery: the illustrious Lebesgue's work contained a *mistake* [GK09, p. 118]!

Lebesgue had made a subtle error—reminiscent in a technical sense of Cauchy's famous mistake a century earlier—in proving that "if E is B-measurable, then its projection is too" [Leb05, p. 191]. His mistake, at root, amounted to failing to notice that the projection of E's complement, need not be the same as the complement of

[29] *Sur la Puissance des Ensembles Mesurables* B.

[30] Here Aleksandrov is taking advantage of the fact that the infinite binary sequences and the real numbers between zero and one are naturally in bijection.

[31] A point set possesses the property of Baire when there is an open set from which it differs—in the sense of symmetric difference—by a set of first category.

E's projection.[32] Suslin, seizing on this, conjectured that a new type of set, perhaps not Borel measurable, might arise. And, critically, its construction would not rely on ordinals or other set-theoretic tools, but rather would be effected by means of the most uncontroversial of all mathematical objects: an explicit continuous function.

Suslin was encouraged in this research by a number of perceived shortcomings of Lebesgue's papers of an entirely different nature. Lebesgue's construction of a non-Borel set depended in an essential way upon the second number class as a *completed entity*:

> "It is know that Mr. E. Borel, following his idea of the *illusion of the transfinite* felt the necessity to address certain criticisms toward [the non-Borel set constructed by Lebesgue] for its employment of the illegitimate totality of transfinite numbers" [Luz25a, p. 1818].

In a 1917 note in *Comptes Rendus* entitled *On a Definition of the* B-*Measurable Sets without Transfinite Numbers*,[33] Suslin offered a simple characterization of the projection of the Borel set set that had eluded Lebesgue:

> "Consider a system S of closed intervals designated by the general notation $\delta_{n_1 n_2 \ldots\, n_k}$, the whole numbers k, n_1, ..., n_k taking all possible values... We say that *a point x belongs to the system S* if there exists a series of positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots$ such that the point x is contained in all of the intervals δ_{α_1}, $\delta_{\alpha_1 \alpha_2}$, ..., $\delta_{\alpha_1 \alpha_2 \ldots \alpha_k}, \ldots$" [Sus17, p. 88-9].

The sets capable of being formed in this way—which Suslin termed the A-sets[34]—included the open intervals, and was closed under countable unions and complements, all of the ingredients necessary to form the Borel sets.

The natural next question, therefore, was, is there an A-set that is not Borel? Suslin was able to show that there is, stating his discovery in the following illuminating fashion: "Theorem II. – There exists a well defined set (in a sense of the word *defined* both logical and precise) that is not a B-measurable set" [Sus17, p. 90]. What is more, he discovered that if both a set E and its complement were A-sets, then E was a Borel set. And a simple analysis showed that the projections of Borel sets, which had led Lebesgue astray, were none other than the A-sets [Sus17, p. 91].

Indeed, much more was known: in an accompanying note, Luzin remarked that Suslin's discovery implied that there exists a function whose range on the unit interval was a non-Borel set. Indeed, the range of any function of the Baire hierarchy has, as its range, an A-set [Luz17, p. 93]. Moreover, the A-sets were all Lebesgue measurable, had the property of Baire—i.e., differed from an open set by a set of first category—and, much as Aleksandrov had shown for the Borel sets, either contained a perfect set or were countable [Luz17, p. 94].

Suslin's discovery, therefore, represented a considerable advance for Luzin's program. It removed the need for ordinals both in formulating the Borel hierarchy and

[32] That is, for example, $p(\mathbb{R}^2 - E)$ need not equal $\mathbb{R} - p(E)$, where $E \subset \mathbb{R}^2$, and $p : \mathbb{R}^2 \to \mathbb{R}$ is given by projection onto the first coordinate.

[33] *Sur une Definition des Ensembles Measurables* B *sans Nombres Transfinis.*

[34] Suslin's nomenclature was the source of an interesting priority dispute. Aleksandrov, perhaps still sour from his failure to solve Luzin's problem, would claim after Suslin's death that the "A" stood for "Aleksandrov," arrogating to himself the discovery of the analytic sets. Actually, Suslin chose A simply because it stand immediately prior to B in the alphabet [GK09, 119].

in proving the existence of a non-Borel measurable set. Moreover, Luzin maintained, the analytic sets uncovered by Suslin held mathematical importance because they "solve, in an exact manner, the problem of Mr. Borel" [Luz25c, p. 1318].

The projective sets

Tragically, Suslin perished in a devastating typhus epidemic that struck Moscow in 1919 [OR11]. But the nascent subject of descriptive set theory, propelled by Suslin's discovery, was becoming a hotbed of activity.

Luzin and his Polish colleague Wacław Sierpiński (1882-1969) turned their attention to co-analytic sets,[35] and soon after, to the so-called *projective sets* [Kan12, p. 419]. While Suslin and Luzin had initially been led to the analytic sets after noticing that they were the projections of Borel sets, "Suslin and I did not attribute any special importance to this proposition, seeing it as more curious than useful" [Luz25b, pp. 1573-4]. But it soon became clear that the projection operation was the decisive factor in producing new and more complicated sets. By alternating the operations of projection and complementation, Luzin observed that one could produce at each stage *new* sets. Following in the footsteps of Lebesgue, Luzin showed in 1925 that the hierarchy was proper, but struggled to establish the regularity properties that had held at lower levels: measurability, the property of Baire, and the perfect set property. Indeed, they were unable to establish the perfect set property even for the co-analytic sets, and for the $PCPE$-sets,[36] every property—measurability, the perfect set property, the property of Baire—seemed out of reach [Luz25a, p. 1819].

Observing the growing difficulties of climbing the new heights, Luzin prophetically announced in a 1925 note in *Comptes Rendus*:

> "The theory of analytic sets presents a perfect harmony: any analytic set is either denumerable or has the cardinality of the continuum; no analytic set is ever a set of third category ... and finally, any analytic set is always measurable. *There is but one important lacuna: we do not know if any uncountable co-analytic set ... has the cardinality of the continuum.* The efforts I've made to resolve this question have led me to this completely unexpected result: there exists a family admitting an application in the hierarchy of effective sets such that one does not know *and one will never know* if every set of this family (suppose uncountable) has the cardinality of the continuum, whether or not it is third category, nor whether it is measurable" [Luz25b, p. 1572].

While it would be decades before Luzin's prognostication would be proven true, it became clear much sooner that it would not be the eager students of Luzitania who would investigate it. Already by 1925, Luzin had begun to shift his attention, spending more time working on his monograph of the theory of functions and less time with undergraduates [OR99]. By 1931, he had shifted his attention entirely to differential equations, and would devote the rest of his life to the pure analysis which had initially led him to descriptive set theory: functions of a real variable, especially problems arising from differential equations [Lav74, p. 173], [Kuz74, p. 202].

[35] I.e., sets whose complements are analytic.

[36] That is, in modern notation, the $\mathbf{\Sigma}^1_2$ sets.

Moreover, Luzin was a man of extremes in his relationships with others, and bitterly resented the students of Luzitania who had moved on to subjects other than descriptive set theory [GK09, p. 186]. He had made enemies of Kolmogorov and Aleksandrov in particular, and in 1936, found himself the subject of a vicious propaganda attack by his political enemies [OR99], [GK09, pp. 153, 160]. His career would never recover after the "Luzin incident," which marked, by most accounts, the end of the era during which soviet scientists could publish abroad [GK09, p. 160]. With the patriarch of Russian set theory subdued by the revolutionary forces that he had feared so greatly in his youth, descriptive set theory entered a period of dormancy.

3.3.4 The limits of mathematics

This dormancy was not just a result of politics, however. Luzin shared in most of the attitudes of his French predecessors and sought to fill in the theoretical gaps they had left. Indeed, if anything, in this regard Luzin was *more* thoroughgoing: the strictures of effectiveness his school observed surpassed those of the French trio. This heightened concern for effectiveness stemmed no doubt from another quality he shared with Baire, Borel, and Lebesgue: a profound concern for the establishment of mathematical truths on the grounds of normal mathematical practice. It was in this spirit that he wrote:

> "The only way of establishing the validity of Cantor's conjecture would be to give a one to one and effective correspondence Z (that is, quite explicitly described, without any indefiniteness or ambiguity) between the points of a straight line on the one hand, and the transfinite numbers of the second class on the other" [Kel74, qtd.,p. 186].

At the same time, Luzin maintained that no such solution would ever be forthcoming [Kel74, qtd., p. 186].

It was significantly in this regard that Luzin saw further than Borel, Baire, or Lebesgue. That is, Luzin sought not only to *expand* mathematical knowledge, but also to identify its limits. In this way, his concern for legitimate mathematics took a different shape from the French Analysts'. His task, as he saw it, was not two-fold, but three-fold: "[Luzin] regarded it as the aim of set theory either to select among mutually exclusive non contradictory assertions one that is valid, or to establish that the very statement of the problem has no meaning" [Kel74, p. 187]. In particular, it was the third position—meaninglessness—that he took in regard to the continuum hypothesis.

Much of his work was in this spirit: the isolation of various possibilities in descriptive set theory, each of which was incompatible with the others, and none of which could be proven. Luzin achieved this not only in the case of the measurability, the Baire property, and the perfect set property, but also, for instance, in the construction of sieves and other natural questions about effective sets [Kel74, p. 182]. It was results of this sort that Luzin thought "compel us to renounce the traditional view of the meaning of the phrase 'the solution of the problem' " [Kel74, qtd., p. 182].

At the same time, Luzin was not entirely pessimistic. He acknowledged, for instance, that

> "One should not forget that Zermelo's argument would lose much of its dubiousness and perilousness if new mathematical facts could be derived from it, establishing new connections between long familiar notions, or at the very least strikingly new and rich insights" [Kel74, qtd., p. 188].

Its damning trait was, rather, its *ad hoc* purpose: to place the cardinality of the continuum *somewhere* in the procession of alephs [Kel74, qtd., p. 188].

Late in the 1920's, Luzin began to follow the developments of Hilbert's school of metamathematics. Holding onto a belief, in the spirit of his French predecessors, that mathematics could not be reduced to a symbol game, he nevertheless rejected the idea that its consistency results could, for instance, resolve the continuum hypothesis [Kel74, p. 188]. Nevertheless, he recognized its "supreme importance," and followed it with great interest [Kel74, p. 188].

Why Luzin made his predictions is something of a mystery. Luzin himself knew no formal logic, nor did he have it in mind when he surveyed the field of descriptive set theory [Keldysh, 188; 179]. Infinitely more mysterious, however, is that he was *right*—and at that, in essentially every regard. Luzin, had not only predicted that the projective sets would stymie mathematics, but also who would bear his prediction out. But it is perhaps appropriate that Luzin, intimately connected as he was to the mathematical traditions of the past, was at the same time fixing the shape that set theory would take in the latter half of the 20^{th} century and into the present.

Chapter 4

Conclusion: Set Theory Modern and Historical

Set theory in the present era faces a difficult problem. Luzin strove to feel out the contours of what it is impossible to know, but his efforts were limited by the lack of a conception *of* his mathematical tools as precise as the theorems he proved *with* them. But this robust metamathematics was not long in coming. While Luzin himself knew little formal logic, the techniques of that branch of mathematics were already ascendant in his time. Some of the earliest independence results—such as Fraenkel's proof of the independence of the axiom of choice from a system of set theory with urelements in 1922 [Fra67, pp. 284-9]—occurred at the same time as the investigations that were leading Luzin to suspect that certain questions in set theory were unanswerable.

Before long, Luzin's premonitions became a fundamental feature of the set theoretic landscape. By 1938, Kurt Gödel (1906-1978) had discovered the constructible universe of sets, and used it to demonstrate the consistency of the axiom of choice and the continuum hypothesis. The same tools demonstrated the possibility of a catastrophic failure of the regularity properties precisely beyond the boundary to which Luzin and the students of Luzitania had pushed them: from the hypothesis that $V = L$, it follows that there is a co-analytic set which has neither the property of Baire nor the perfect set property, and a Δ^1_1 set that is not Lebesgue measurable [Kan94, p. 150].

After Gödel's delimitative result, however, matters seemed, for a time, to have reached another impasse. It was in 1963, more than two decades later, when Paul Joseph Cohen (1934-2007) announced the resolution of the then nearly one hundred year-old problem of the continuum hypothesis: it is independent of **ZFC** [Kan94, p. 114]. With that, the floodgates opened. Cohen's forcing technique enabled set theorists to demonstrate the independence of numerous long-standing open problems. Within two years, Robert Martin Solovay (1938-) used a combination of forcing and inner models to demonstrate the antithesis of Gödel's earlier result, namely, that it was consistent with **ZFC** (and the existence of an inaccessible cardinal) that *all* subsets of the real line are measurable, have the property of Baire, and have the perfect set property [Sol70, p. 1].

The upshot is that the projective sets are, in a strong sense, *maximally* uncalibrated in **ZFC**. This is an example of what William Hugh Woodin (1955-) has termed the "**ZFC** Dilemma": many, if not most, of the fundamental questions of set theory are left completely open by **ZFC** [Woo11d, p. 449]. The dilemma is philosophically unsatisfying, but, more to the point, it is mathematically unsatisfying. In light of it, one cannot, it would seem, make further progress on questions formerly central to the set-theoretic enterprise.

But there is an alternative. Independence is a phenomenon that occurs relative to a given axiom system; the issue, one might argue, lies not with set theory itself but rather with **ZFC**. Set theory has never ceased to be a controversial mathematical subject, and recently the debate has turned to a new question: does mathematics need new axioms?

Such a question requires unpacking. As Solomon Feferman points out,

> "The question, 'Does mathematics need new axioms?,' is ambiguous in practically every respect. What do we mean by 'mathematics'? What do we mean by 'need'? What do we mean by 'axioms'? You might even ask, What do we mean by 'does'?" [Fef99, p. 99].

Various ways of disambiguating are, of course, available. But I think that one of the most illuminating is, rather than merely asking "How *should* one define 'need,' 'mathematics,' and 'axioms'?" to inquire instead "How *have* mathematicians defined 'need,' 'mathematics,' and 'axioms'?"

The jumping-off point for the present essay was the following question: would Borel, Baire, Lebesgue, and Luzin see as legitimate—or even *mathematical*—the modern resolution to the problem of **PD**? At this point, there is little doubt that the answer would be "no": those for whom the existence of $\aleph_1$ is suspect are certain to give no credence to the staggering transfinite heights upon which set theorists such as Woodin and John Robert Steel (1948-) base their arguments[1] for the truth of **PD** [FFMS00, pp. 422-431].

Nevertheless, as we have seen, so too would Archimedes have rejected the infinitary methods of Newton and Leibniz; what is more, at least from a modern perspective, he would have been far more justified in rejecting the calculus—which, in the 17^{th} and 18^{th} centuries wore its inconsistencies almost as badges of honor[2]—than any of Borel, Baire, Lebesgue, or Luzin would be in rejecting transfinite objects which, even after almost a century of systematic research, have not succumbed to inconsistency.

But in spite of its good press, consistency has played much less of an important role in determining what the mathematical community accepts or rejects as legitimate mathematics than is frequently assumed. Rather, as I have argued, the determiners of legitimacy for new mathematics are above all usefulness in advancing *already established* mathematical programs and the possibility of demarcating a safe domain for its use. Cantor's set theory met with resistance in large part because it was not clear, at least initially, how it contributed to the various currents of late 19^{th} century mathematics, and because of the immense difficulty of determining when and how it could safely be used; Borel, Baire, Lebesgue, and, to a lesser extent, Luzin remedied the situation by showing what part of Cantor's theory *could* safely

[1] I do not mean to impute to Woodin and Steel an ungenerous and simple reading of their argument such as, e.g., that because there are (sufficiently) large cardinals, **PD** is true. Rather, my point is that it is unlikely that Borel, Baire, Lebesgue, or Luzin would have entertained *any* sort of argument involving large cardinals.

[2] Recall, for instance, Leonard Euler's comfort with summing $1 + 2 + 4 + \ldots$ to -1.

be used, and for *which* mathematical purposes. Their attitude toward set theory admittedly lacked a certain degree of philosophical principle, but it was a compromise that was enormously successful mathematically.

One could also point to how precisely the opposite fact pattern obtained during the rise of the calculus. It was immediately evident to what mathematical problems the techniques of Newton and Leibniz could fruitfully applied, and also immediately evident, as Berkeley observed, that when one "attentively [considered] the things themselves, ... we shall discover much Emptiness, Darkness, and Confusion; nay, if I mistake not, direct Impossibilities and Contradictions" [Ber34, p. 4]. But, even Berkeley conceded that "It must indeed be acknowledged [that] the modern Mathematicians do not consider these Points as Mysteries, but as clearly conceived and mastered by their comprehensive Minds" [Ber34, p. 3]. This stemmed no doubt from the ease of cutting out a safe domain, even without resolving the foundations of the new infinitary techniques. Recall that Euler, for instance, was well aware of the potential of his method of moving between the expressions of functions and their series expansions without regard to convergence to generate "paradox and unnatural results, [but] his great success with the general method liscensed by his principle ... took precedence over his qualms" [Lis00, p. 259].

Of course, the question of determining *what* constitutes a successful mathematical application is an exceptionally thorny one, and one which figures prominently in contemporary debate over the axioms of set theory. One might like explicitly capturing the notion of mathematical success to to the equally difficult philosophical task of accurately codifying the notion of justice in a system of laws. The difficulty no doubt contributes greatly to the second prong of my argument, namely, that in practice, mathematicians take the common law approach, and base their opinion on what constitutes *valuable* or *legitimate* mathematics on historical precedent.

It is for this reason that I imagine Harvey Friedman (1948-) may have the last word on the adoption of new axioms into mathematics. He argues that the circumstances that would have to surround the adoption of new axioms by the mathematical community would require that any new axiom be mathematically natural, concrete, have "points of contact with a great variety of mathematics," and, critically, be "used in normal mathematics" [FFMS00, pp. 438-440]. He has deduced, correctly in my opinion, that mathematics pushes forward always while looking rearward. If the program of overcoming independence is successful, it will be through forging a connection to the heritage modern set theory shares with the rest of mathematics.

Part II

A Mathematical Approach

Chapter 5

Large Cardinals and Infinite Games

5.1 Introduction

The study of games of perfect information has its roots in the works of Zermelo, who in 1913 argued that if a player in a game of chess can force a win at all from a certain position, then she must be able to force a win in some finite number of moves. While his proof was incomplete, the essential gap was filled by Dénes König in 1927 using his famous tree lemma. Borel also studied the general concept of a strategy, and formulated the notion of *minimax* in 1921. With Von Neumann's proof of the Minimax Theorem in 1928, the theory of games was fully formed [Kan94, p. 371].

The interaction between games of perfect information and topological spaces began with the investigations of Banach, Mazur, and Steinhaus in the 1920's and 1930's [Kan94, p. 371]. The importance of infinite games for modern set theory only became apparent, however, following the work of Mycielski and Steinhaus in the 1960's [Lar12, p. 458]. Crucially, a surprising connection was observed between *determinacy*—that is, one of the two opposing players having a winning strategy—and the regularity properties which formed the heart of descriptive set theory. Mycielski and Steinhaus observed in 1962 that if all games played on sets of reals are determined, then every set is measurable, has the property of Baire, and has the perfect set property. The purpose of this hypothesis, which they termed the *Axiom of Determinacy* (**AD**), was not to "depreciate ... classical mathematics ... but only to propose another theory which seems very interesting, although its consistency is problematic" [Lar12, qtd., p. 466].

While the work of Gale and Stewart in the 1950s has already shown that, using the axiom of choice, it is possible to create a set of reals which is *not* determined, they also showed that open sets *are* determined and so interest developed in exploring how complex sets that are determined can become. Early progress was made by Martin, who by 1974 had shown that all Borel sets are determined. At the same time, the perspective of infinite games was yielding unexpected connections to large cardinals—including the notable early work of Solovay, who established that under **AD**, ω_1 is a measurable cardinal—and allowing new proofs of classical theorems in descriptive set theory [Lar12, p. 459].

A tight intertwining of large cardinals and determinacy hypotheses soon emerged, coalescing especially around the search to determine the precise consistency strength of the various levels of the projective hierarchy. While it was ultimately established that $\mathbf{ZF} + \mathbf{AD}$ is equiconsistent with the existence of infinitely many Woodin cardinals in **ZFC**, early forays into the projective hierarchy seemed to suggest a very different story [Kan94, p. 466]. All initial evidence pointed to the necessity of measuring determinacy's strength by large cardinal hypotheses verging on the Kunen inconsistency.

The following two chapters are intended, therefore, as a mathematical reconstruction of those events. The present chapter is divided into three sections: the regularity properties which initially drew interest to **AD** in Section 5.2, Martin's proof of Borel determinacy in 5.3, and the initial interaction between the theory of large cardinals and determinacy in Section 5.4. Chapter 6, where we present the heretofore unpublished first proof of projective determinacy from a large cardinal axiom, is the culmination of this story.

5.1.1 Preliminaries

We assume the reader is familiar with the basic notions of logic and descriptive set theory, such as elementary embeddings, trees, measurability (by which we mean, unless otherwise indicated, *Lebesgue* measurability), the Baire property, and so on. Suitable background references include [Kec95], [Mos09], and Chapter 11 of [Jec03]. The reader will also require familiarity with forcing and the constructible universe for Chapter 6. Chapters 13 and 14 in [Jec03] contain all of the necessary background material.

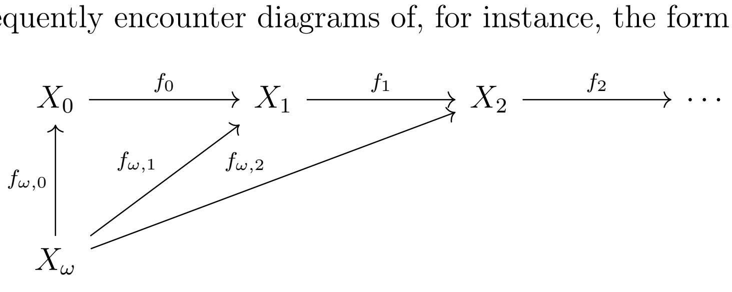
We shall maintain the following notational conventions:

(i) For sets A and B, ${}^{A}B$ denotes the set of functions from A to B; we reserve the notation A^B for cardinal arithmetic. Likewise, $f^{\text{PRE}}(A)$ denotes the preimage of A by f; f^{-1} is left for invertible maps. The restriction of a function $f : A \to B$ to a subset A' of A is denoted $f\restriction_{A'}$.

(ii) The Baire space ${}^{\omega}\omega$ will be denoted by $\mathcal{N}$, and the Cantor space ${}^{\omega}2$ will be denoted by $\mathcal{C}$.

(iii) If T is a tree and $t \in T$, then $|t|$ denotes the lenght of t. Two nodes t, s are said to be *compatible* if either t extends s or s extends t. We denote by $[T]$ the space of all of T's infinite branches, with the topology generated by the clopen sets $O(t) = \{x \in [T] : x\restriction_{|t|} = t\}$, and by T_s the set of all $t \in T$ compatible with s. By convention, we assume that trees on $\omega \times \omega$, i.e., trees formed with nodes in the set ${}^{<\omega}\omega \times {}^{<\omega}\omega$ are assumed, for simplicity, to consist of pairs of sequences (s, t) where $|s| = |t|$.

(iv) We shall frequently encounter diagrams of, for instance, the form



(5.1)

or possibly with the arrows reversed. We shall always assume that $f_i : X_i \to X_{i+1}$. We write $f_{i,j}$ where $i, j \leq \omega$ is the unique map starting at the i-indexed object and ending at the j-indexed object. Consequently, $f_{i,i+1} = f_i$.

(v) We use angle brackets $\langle -, - \rangle$ and parentheses $(-, -)$ interchangeably to indicate finite sequences; however, we always use $\langle x_\beta : \beta < \alpha \rangle$ to denote a β-sequence, i.e., a map from β to $\{x_\alpha : \alpha < \beta\}$. For a finite sequence of elements $(x_0, \ldots, x_n)$, we will occasionally adopt the shorthand $\vec{x}$.

(vi) For a topological space X and $\alpha < \omega_1$, $\mathbf{\Pi}^0_\alpha[X]$ and $\mathbf{\Sigma}^\alpha_0[X]$ denote the obvious segments of the Borel hierarchy on X. When we omit X, as in $\mathbf{\Sigma}^0_\alpha$, X is assumed to be $\mathcal{N}$. Similarly, we denote the projective subsets by $\mathbf{\Sigma}^1_0$, $\mathbf{\Pi}^1_0$, $\mathbf{\Sigma}^1_1$, etc.

(vii) For an elementary embedding $j : V \to V$ or $j : V_\delta \to V_\delta$, we denote by $\textsc{Crt}(j)$ the least ordinal α such that $j(\alpha) > \alpha$. For an ordinal α, $\textsc{Cof}(\alpha)$ denotes its confinality. We denote by $\textsc{Form}$ the set of all formulae; by convention $\textsc{Form} \in V_{\omega+1}$. In general, boldface letters such as $\mathbf{x}$ and $\mathbf{y}$ are reserved for formal variables as in, e.g., $\varphi(\mathbf{x}, \mathbf{y}) \in \textsc{Form}$. The notation $\textsc{Ot}(R)$ denotes the order type of R.

5.2 Infinite Games and Regularity Properties

In this section, we follow the presentation of [Kan94, pp. 368-377].

5.2.1 Notation

Let X be a non-empty set. Then, for $A \subset {}^\omega X$, the *standard game* $G_A(X)$ describes the following situation: two players, Player I and Player II, take turns playing elements of X, as in the following schematic:

$$G_A(X)$$

I	II
x_0	
	y_0
x_1	
	y_1
x_2	
	y_2
$\vdots$	$\vdots$

(5.2)

Player I is said to *win* when the sequence $(x_0, y_0, x_1, y_1, \ldots) \in A$; otherwise Player II wins. The set A is called the *payoff set.* A finite sequence

$$p = (x_0, y_0, \ldots, x_n, y_n) \tag{5.3}$$

is called a *partial play with Player I to play*, and a finite sequence

$$p = (x_0, y_0, \ldots, x_n) \tag{5.4}$$

is called a *partial play with Player II to play.* A partial play is also called a *position.* An infinite sequence $p \in {}^\omega X$ is simply called a *play* of the game. We denote the set of all partial plays with Player I to play by $\mathscr{P}_{\mathrm{I}}$ and the set of all partial plays with Player II to play by $\mathscr{P}_{\mathrm{II}}$. Similarly, $\mathscr{P}_\omega$ denotes the set of all infinite plays.

For brevity, for a partial or infinite play p we denote by p_{I} the even-indexed entries of p—i.e., what would have been played by Player I—and by p_{II} the odd-indexed entries of p—i.e., what would have been played by Player II.

If $x = (x_0, x_1, x_2, \ldots) \in {}^\omega X$ and $y = (y_0, y_1, y_2, \ldots) \in {}^\omega X$, then we set $x * y = (x_0, y_0, x_1, y_1, \ldots)$, i.e., the play of the game that would result if Player I played according to x and Player II according to y.

A *strategy* for Player I is a map $\sigma : \bigcup_{n<\omega} {}^{2n}X \to X$; a strategy τ for Player II is a map $\tau : \bigcup_{n<\omega} {}^{2n+1}X \to X$. We say for a play p that Player I *played according to the strategy* σ when, setting $y = p_{\mathrm{II}}$ and $x = p_{\mathrm{I}}$, for all $n < \omega$, $x_n = \sigma(x_0, y_0, \ldots, y_{n-1})$. Conversely, we say that Player II played according to τ when $y_n = \tau(x_0, y_0, \ldots, x_n)$ for all $n < \omega$. We denote by $\sigma * y$ the play of the game where Player II plays y and Player I plays against y by the strategy σ, and similarly, $x * \tau$ denotes the play of the game where Player I plays x and Player II plays against x by τ. For partial plays p, $p * \sigma \in \mathscr{P}_{\mathrm{II}}$ and $\tau * P \in \mathscr{P}_{\mathrm{I}}$ are defined analogously. We let $\mathscr{S}$ denote the set of strategies, both for Player I and Player II.

We call a strategy σ a *winning strategy for Player I* when for all $y \in {}^\omega X$, $\sigma * y \in A$. Likewise, a strategy σ is a winning strategy for Player II when $x * \sigma \notin A$ for all $x \in {}^\omega X$.

A set A is said to be *determined* if either Player I or Player II has a winning strategy.

Let T be a pruned tree on ${}^\omega X$. Then, can also consider the game $G_A(T)$, where T is said to define the *legal positions*, which is the same as $G_A(X)$ except that every partial play is required to be in T—the first player to play an element of X such that the resulting position is not in T loses. Note that this is merely a notational convenience: the win and loss conditions of $G_A(T)$ are identical to those of $G_B(X)$, where B is the set

$$B = \{x \in {}^\omega X : (\forall n < \omega)(x \in A \wedge x\restriction_{2n} \notin T \to x\restriction_{2n-1} \notin T) \\ \vee (\exists n < \omega)(x\restriction_{2n} \in T \wedge x\restriction_{2n+1} \notin T)\}. \quad (5.5)$$

We shall also explore other kinds of games besides the standard game. In general, the notions of a strategy, a play, a payoff set, etc. all generalize in obvious ways, and we shall adapt our notation and terminology accordingly.

5.2.2 Basic results

Theorem 5.2.3. *If A is an open subset of T, then the game $G_A(T)$ is determined.*

Proof. The trick is to notice that membership in an open set is determined at some finite stage of the game. That is, for all $x \in T$, $x \in A$ if and only if $O(x\restriction_n) \subset A$ for some n, where $O(s) = \{t \in T : t \text{ extends } s\}$. (The sets $O(s)$ for $s \in T$ form a basis of the tree topology for $[T]$.)

It suffices to show that if Player I does not have a winning strategy, then Player II does. Call a position p *not winning for Player I* if Player I does not have a winning strategy in the game $G_{A/p}(T/p)$, where $A/p = \{t \in {}^\omega X : p^\frown t \in A\}$ and $T/p = \{s \in {}^{<\omega} X : p^\frown s \in T\}$.

Now, suppose Player I does not have a winning strategy. Then, for every $x_0 \in X$, there is some $y_0 \in X$ such that Player I does not have a winning strategy at (x_0, y_0). Therefore, let $\tau(x_0)$ equal this y_0. Likewise, we set $\tau(x_0, y_0, x_1)$ to be some y_1 such that (x_0, y_0, x_1, y_1) is not winning for Player I, and so on. Then, it follows that $\vec{x} * \tau$ for any $\vec{x} = (x_0, \ldots, x_n)$ is not a winning position for Player I.

We claim that σ is a winning strategy for Player II. For, suppose that $p = x * \tau \in A$. Then, there exists some n such that $O(p\restriction_n) \subset A$. Without loss of generality, suppose n is odd. Then $p\restriction_n = (p\restriction_n)_{\mathrm{I}} * \tau$ is a winning position for Player I, contrary to hypothesis. □

Theorem 5.2.4. *Assume* **AC**. *Then there is a set of reals A such that A is not determined.*

Proof. Note that the set $\mathscr{S}$ of strategies has the same cardinality as $\mathcal{N}$, namely, $2^{\aleph_0}$. Let $\langle \sigma_\alpha : \alpha < 2^{\aleph_0} \rangle$ be an enumeration of the strategies for Player I, and $\langle \tau_\alpha : \alpha < 2^{\aleph_0} \rangle$ an enumeration of the strategies for Player II. Then, we shall construct two disjoint sets A and B by diagonalizing against the strategies.

More specifically, define a_α and b_α inductively. Supposing that a_β and b_β have already been chosen for $\beta < \alpha$, note that since $|\{a_\beta : \beta < \alpha\}| < 2^{\aleph_0}$, there exists some $y \in \mathcal{N}$ such that $\sigma_\alpha * y \notin \{a_\beta : \beta < \alpha\}$; then, set $b_\alpha = \sigma_\alpha * y$. Similarly, we set a_α to be $x * \tau_\alpha$, where x is chosen so that $a_\alpha \notin \{b_\beta : \beta < \alpha\}$. Let $A = \{a_\alpha : \alpha < 2^{\aleph_0}\}$, and define B similarly.

It is clear that A is not determined. For, if σ_α is any strategy for Player I, then, by construction, $\sigma_\alpha * y \in B$ for some $y \in \mathcal{N}$. The same argument shows that Player II has no winning strategy. □

5.2.5 Regularity properties

Definition 5.2.6 (The Banach-Mazur Game)**.** Consider the following game $G^{**}_\omega(A)$, played in the following way: Players I and II alternate choosing finite sequences, as indicated in the schematic below:

$$G^{**}_\omega(A)$$

I	II
s_0	
	t_0
s_1	
	t_1
s_2	
	t_2
$\vdots$	$\vdots$

(5.6)

Let $x = s_0{}^\frown t_0{}^\frown s_1{}^\frown t_1{}^\frown \ldots$. Then, Player I wins if $x \in A$, and Player II wins if $x \notin A$. The game $G^{**}_\omega(A)$ is called the *Banach-Mazur Game*.

Lemma 5.2.7.

(i) If Player II has a winning strategy in $G^{**}_\omega(A)$, then A is a meager subset of $\mathcal{N}$;

(ii) If Player I has a winning strategy in $G^{**}_{\omega}(A)$, then A is co-meager relative to the open set $O(s_0)$.

Proof. We begin with Case (i). First, suppose that A is meager, so that $A \subset \bigcup_{n<\omega} F_n$, where the F_n is a nowhere dense closed set for each $n < \omega$. Then, we seek a strategy τ for Player II. Suppose that Player I plays s_0 on her first turn. Then, since F_0 is nowhere dense, there exists some t_0 such that $O(s_0^\frown t_0)$ is disjoint from F_0. Set $\tau(\langle s_0 \rangle) = t_0$. Likewise, suppose that $p = \langle s_0, t_0, \ldots, s_n \rangle$ is some partial play with Player II to play; then, since F_n is nowhere dense, there exists some sequence t_n such that $O(s_0^\frown t_0^\frown \ldots ^\frown s_n^\frown t_n)$ is disjoint from F_n. Set $\tau(p) = t_n$. It is clear that this is a winning strategy for Player II, since Player II prevents x from being an element of F_n on her n-th turn.

Next, suppose that Player II has a winning strategy. For each partial play $p = \langle s_0, t_0, \ldots, s_n, t_n \rangle$, let p_* be the sequence $s_0^\frown t_0^\frown \ldots ^\frown s_n^\frown t_n$. We can define the set

$$D_p = \Big\{ x \in \mathcal{N} \ : \ \text{if } x \in O(p_*), \text{ then for some } t,\ x \in O\big(p_*^\frown t^\frown \tau(p^\frown \langle t \rangle)\big) \Big\}, \tag{5.7}$$

i.e., D_{p_*} is the union of $-O(p_*)$ (which is clopen) and those x in $O(p_*)$ where x is still a possible play of the game after Player I responds on her turn by some t. Clearly D_{p_*} is open, since if $x \in D_{p^*} \cap O(p_*)$, then $O(x{\restriction}_m) \subset D_{p_*} \cap O(p_*)$, where $x{\restriction}_m = p_*^\frown t^\frown \tau(p^\frown \langle t \rangle)$ for some t. Density follows similarly, once we note that for any sequence t, $O(p_*^\frown t^\frown \tau(p^\frown \langle t \rangle)) \subset O(p_*^\frown t)$. Therefore, since $|\mathscr{P}_{\mathrm{I}}| = \aleph_0$, for all x such that

$$x \in \bigcap_{p \in \mathscr{P}_{\mathrm{I}}} D_p \tag{5.8}$$

by recursively applying Equation 5.7, we see that there is some infinite play p such that

$$x = s_0^\frown t_0^\frown s_1^\frown t_1^\frown \ldots \tag{5.9}$$

where $p * \tau = \langle s_0, t_0, \ldots \rangle$. Therefore $x \notin A$, so

$$S \subset \bigcap_{p \in \mathscr{P}_{\mathrm{I}}} (\mathcal{N} - D_{p_*}) \tag{5.10}$$

and so A is meager.

To show Case (ii), simply note that if Player I initially plays s_0, then we can consider Players I and II as playing $G^{**}_{\omega}(-(A/s_0))$ with their roles reversed. That is, Players I and II play according to the schematic below:

$G^{**}_{\omega}(-(A/s_0))$

I	II
	t_0
s_1	
	t_1
s_2	
	t_2
s_3	
$\vdots$	$\vdots$

(5.11)

(In the new game, Player II plays first, and Player I plays second.) Then, Player I's winning strategy carries over, giving her a winning strategy for $G^{**}_\omega(-(A/s_0))$. Therefore, by the proof of Case (i), $-(A/s_0)$ is meager; hence, $O(s_0) - A$ is meager, as desired. □

Definition 5.2.8. Let X be an arbitrary topological space, and let $A \subset X$. Define the *canonical open set associated to* A to be where

$$O_A = \bigcup \{U \subset X : U \text{ is open and } A \text{ is co-meager relative to } U\}. \tag{5.12}$$

Lemma 5.2.9. *Assume* **AC**. *$O_A - A$ is meager. Moreover, a set A has the Baire property if and only if $A - O_A$ is meager.*

Proof. Let $\mathcal{U} = \{U \text{ open} : A \text{ is co-meager in } U\}$. Let $\{W_i\}_{i \in I}$ be a maximal family of disjoint elements of U. Then $W = \bigcup_{i \in I} W_i$ is open and dense in O_A, so $O_A - W$ is meager. Since A is co-meager in each W_i and the W_i are disjoint, A is co-meager in W. Thus,

$$O_A - A \subset (O_A - W) \cup (A - W) \tag{5.13}$$

so $O_A - A$ is meager.

Now, suppose A has the Baire property, so that $U \triangle A$ is meager for some open set U. Then, trivially, $U - A$ is meager, so $U \subset O_A$, and hence $A - O_A \subset A - U$, and the latter is meager. □

Remark 5.2.10. While the proof of Lemma 5.2.9 made use of the full axiom of choice, so long as X is second countable, we may assume that $\mathcal{U}$ is the collection of basic open sets in which A is co-meager. Then, we only require the axiom of countable choice to construct $\{W_i\}_{i \in I}$. This is relevant in light of Theorem 5.2.4. However, observe that the proof of Theorem 5.2.4 *cannot* be carried through using **DC** or **CC**.

Theorem 5.2.11. *A set $A \subset \mathcal{N}$ has the Baire property if and only if the game $G^{**}_\omega(A - O_A)$ is determined.*

Proof. Only Player II can have a winning strategy in $G^{**}_\omega(A - O_A)$. For, if Player I had a winning strategy, then $O(s) - (A - O_A)$ would be meager for some sequence s by Lemma 5.2.7. Then, $O(s) - A$ would be meager, so $O(s) \subset O_A$. Then, it follows that $O(s) - (A - O_A) = O(s)$, which is impossible, since non-empty open subsets of $\mathcal{N}$ are not meager by the Baire Category Theorem.

Now, if Player I has a winning strategy, then $A - O_A$ is meager again by Lemma 5.2.7, so by Lemma 5.2.9, A has the Baire property. □

Remark 5.2.12. Theorem 5.2.11 is actually a special case of a more general result. As indicated in the proof of Lemma 5.2.7, one may think of Players I and II as playing a sequence of open subsets of a topological space X—without loss of generality, these subsets can be assumed part of some collection of basic open sets—in the following way:

$$G_A^{**}(X)$$

$$\begin{array}{c|c}
\text{I} & \text{II} \\
\hline
U_0 & \\
 & V_0 \\
U_1 & \\
 & V_1 \\
U_2 & \\
 & V_2 \\
\vdots & \vdots
\end{array} \tag{5.14}$$

Here, $U_0 \supset V_0 \supset U_1 \supset U_2 \supset \dots$, and Player I wins if $\bigcap_{n<\omega} U_n = \bigcap_{n<\omega} V_n$ is a subset of A. Then, the following generalization of Lemma 5.2.7 holds, with essentially the same proof:

1. The set A is meager if and only if Player II has a winning strategy;
2. If X is a Choquet,[1] with a metric $d : X \times X \to \mathbb{R}_{\geq 0}$ generating a coarser topology than the topology on X, then Player I has a winning strategy if and only if A is co-meager relative to a non-empty open set.

(See Theorem 8.33 in [Kec95].)

In particular, the general theorem that A has the Baire property if and only if $G^*(O_A - A)$ is determined remains true.

Definition 5.2.13 (The Davis Game)**.** The *Davis Game* $G_2^*(A)$, for $A \subset \mathcal{C}$, is played as follows: Players I and II play alternately, with Player I choosing elements of ${}^{<\omega}2$, i.e., finite binary sequences, and Player II choosing elements of $2 = \{0, 1\}$, as indicated in the schematic below:

$$G_2^*(A)$$

$$\begin{array}{c|c}
\text{I} & \text{II} \\
\hline
s_0 & \\
 & m_0 \\
s_1 & \\
 & m_1 \\
s_2 & \\
 & m_2 \\
\vdots & \vdots
\end{array} \tag{5.15}$$

Then, set $x = s_0^\frown \langle m_0 \rangle^\frown s_1^\frown \langle m_1 \rangle^\frown \dots$. Player I wins if $x \in A$, and Player II wins if $x \notin A$.

Theorem 5.2.14. *Let* $A \subset {}^\omega 2$.

[1]Recall that a topological space X is Choquet if Player I has a winning strategy in the Choquet game, where the Choquet game is the general Banach-Mazur game played on $A = X$.

(i) The set A is countable if and only if Player II has a winning strategy in $G_2^*(A)$;

(ii) The set A contains a perfect subset if and only if Player I has a winning strategy in $G_2^*(A)$.

In particular, A has the perfect set property if and only if $G_2^*(A)$ is determined.

Proof. First, suppose that there is an enumeration $\langle a_i : i < \omega\rangle$ of A. Then, Player II can simply play m_n on her m-th turn so that the resulting sequence differs from the initial segment of a_n of the same length.

Next, suppose that Player II has a winning strategy. Now, in parallel to the proof of Lemma 5.2.7, suppose that $p = \langle s_0, m_0, s_1, m_1, \dots, m_n\rangle$ is some partial play with Player I to play. Let $p_* = s_0^\frown\langle m_1\rangle^\frown s_2^\frown \dots ^\frown\langle m_n\rangle$. Then, as before, set

$$D_p = \Big\{x \in \mathcal{C} : \text{if } x \in O(p_*), \text{ then for some } t,\ x \in O\big(p_*^\frown t^\frown \tau(p^\frown\langle t\rangle)\big)\Big\}. \tag{5.16}$$

As before, we see that $A \subset \bigcup_{p\in\mathscr{P}_{\mathrm{I}}} D_p$. However, in addition to being a dense open set, for each p, $\mathcal{C} - D_p = \{x_p\}$ for some single point x_p. To see this, note that since, for $x_p \in \mathcal{C} - D_p$, $x_p \notin O\big(p_* \frown t \frown \sigma(\langle p_*^\frown t\rangle)\big)$; hence, it follows that $x_p(|p_*|) = 1 - \tau(p_*^\frown\emptyset)$, and in general,

$$x_p(m) = 1 - \tau\big(p_*^\frown\langle x_p(|p_*|)\rangle^\frown\langle x_p(|p_*|+1)\rangle^\frown \dots ^\frown\langle x_p(m-1)\rangle\big). \tag{5.17}$$

Thus x_p is completely determined. Therefore A is countable.

Next, suppose that A contains a perfect subset P. Define

$$T = \{x{\restriction}_n : x \in P \text{ and } n < \omega\}. \tag{5.18}$$

Note that $P = [T]$. Then, since P has no isolated points, it follows that for any $s \in T$, there are t_0 and t_1 such that $s \subset t_0$, $s \subset t_1$, and t_0 and t_1 are incompatible. In particular, we can assume that $|t_0| = |t_1| = m$, and $t_0(m) \neq t_1(m)$, while $t_0(i) = t_1(i)$ for all $i < m$. We call such t_0 and t_1 a *splitting pair*.

Then, we build a strategy for Player I in the following way: assume that p is some partial play of the game with Player I to play, and let p_* be as before. Then, in T, there is some splitting pair $t_0, t_1 \supset p_*$. Suppose that Player I's play is her $(n+1)$-st turn. Then, let Player I play s_n, where s_n is the sequence such that $p_*^\frown s_n = t_0{\restriction}_{m-1} = t_1{\restriction}_{m-1}$. Then it is clear that however Player II plays, the resulting infinite play p will describe a point $x \in P$.

Lastly, suppose Player I has a winning strategy σ. Then the set $\{\sigma * p_{\mathrm{II}} : p \in \mathscr{P}_\omega\}$ is trivially a perfect set, as well as a subset of A. □

Remark 5.2.15. We fix some enumeration of the finite sequences $\mathfrak{s}_0 = \emptyset$, $\mathfrak{s}_1$, $\mathfrak{s}_2$, etc. such that if $s \subset t$, then $s = \mathfrak{s}_i$ and $t = \mathfrak{s}_j$ for $i < j$. We shall occasionally write $\mathfrak{s}(i)$ instead of $\mathfrak{s}_i$ for notational clarity.

We also note that there is a simple homeomorphism Ψ between $\mathcal{N}$ and a G_δ subset of $\mathcal{C}$ consisting of finite sequences which are not eventually constant:

$$\Psi : (n_0, n_1, n_2, \dots) \longmapsto 1\overbrace{00\dots0}^{n_0+1}1\underbrace{00\dots0}_{n_1+1}1\dots \tag{5.19}$$

Definition 5.2.16 (The Harrington Covering Game)**.** For $A \subset \omega$ and $\epsilon \in \mathbb{R}_{>0}$, the *Harrington Covering Game* $G(A, \epsilon)$ is played by having Players I and II alternate playing natural numbers, as indicated in the schematic below:

$$G_A(X)$$

$$\begin{array}{c|c} \text{I} & \text{II} \\ \hline n_0 & \\ & m_0 \\ n_1 & \\ & m_1 \\ n_2 & \\ & m_2 \\ \vdots & \vdots \end{array} \tag{5.20}$$

Player I's moves are legal only if they are all 0's or 1's. Player II's k-th move is legal if, setting

$$N_i = O(\mathfrak{s}_{\mathfrak{s}_i(0)}) \cup \ldots \cup O(\mathfrak{s}_{\mathfrak{s}_i(|\mathfrak{s}_i|-1)}) \tag{5.21}$$

then N_{m_k} has measure less than $\epsilon/2^{2(k+1)}$. Then, Player I wins if the sequence $x = (n_0, m_0, n_1, m_1) \in \Psi(\mathcal{N})$ and $\Psi^{-1}(x_{\text{I}}) \in A - \bigcup_{n<\omega} N_{m_n}$; otherwise, Player II wins.

Lemma 5.2.17. *In the game* $G(A, \epsilon)$*,*

(i) If Player I has a winning strategy then there is measurable set $B \subset A$ *such that* $m(B) > 0$*;*

(ii) Conversely, if Player II has a winning strategy, then there is an open set U *containing* A *such that* $m(O) < \epsilon$*.*

Proof. Suppose Player I has a winning strategy σ. Note that the map $y \mapsto \sigma * y$ is continuous. Hence, the set $\{y * \sigma : y \in \mathcal{N}\}$ is analytic, and therefore measurable; see Theorem 11.18 in [Jec03]. Then, the set

$$B = \{\Psi^{-1}((\sigma * y)_{\text{I}}) : y \in \mathcal{N}\} \tag{5.22}$$

is a measurable subset of A. Moreover, necessarily $m(B) > 0$; otherwise, there would exist basic open sets $O(s_0), \ldots, O(s_n)$ such that

$$B \subset \bigcup_{i=0}^{n} O(S_i) \qquad \sum_{i=0}^{n} m(O(s_i)) < \epsilon/4. \tag{5.23}$$

Then, $\bigcup_{i=0}^{n} O(s_i) = N_j$ for some j, so if Player II plays j on her first turn, she wins, contrary to the assumption that σ was a winning strategy for I.

Next, suppose Player II has a winning strategy τ. Then, for any $s \in {}^{<\omega}2$ of length $n > 0$, let $d(s) = s * \tau$, i.e., Player II's response, according to τ, to the partial play $\langle s(0), \tau(\langle s(0) \rangle), s(1), \ldots, s(n) \rangle$. Then, since τ is a winning strategy, for every $x \in A$, there is some $s \in {}^{<\omega}2$ such that $x \in N_{d(s)}$, i.e.,

$$A \subset \bigcup_{s \in {}^{<\omega}2} N_{d(s)} \tag{5.24}$$

Now, note that there are 2^n different sequences of length n; moreover, since τ is winning, $m(N_{d(s)}) < \epsilon/2^{2n}$ for $|s| = n$. Thus, it follows that

$$m\left(\bigcup_{s\in{}^{<\omega}2} N_{d(s)}\right) \leq \sum_{n=1}^{\infty}\sum_{|s|=n} m(N_{d(s)}) \leq \sum_{n=1}^{\infty}\sum_{|s|=n} \epsilon/2^{2n} = \sum_{n=1}^{\infty} \epsilon/2^n = \epsilon. \tag{5.25}$$

□

Theorem 5.2.18. *Suppose that $B \supset A$ is a measurable superset of* minimal *measure, i.e., if C is some other measurable set and $C \subset B-A$, then $m(C) = 0$. Suppose that for any $\epsilon \in \mathbb{R}_{>0}$ the game $G(B-A,\epsilon)$ is determined. Then A is measurable and $m(A) = m(B)$.*

Proof. By construction, only Player II can win $G(B-A,\epsilon)$. Therefore, for all ϵ, there is an open set U_ϵ of measure less than ϵ such that $B - A \subset U_\epsilon$. Thus, $B - A \subset \bigcap_{0<n<\omega} U_{1/n}$, and the latter is a set of measure zero. Therefore $B - A$ has measure zero, so A is measurable also, and $m(A) = m(B)$. □

Definition 5.2.19 (The Axiom of Determinacy)**.** We say that the Axiom of Determinacy (abbreviated **AD**) holds if for all $A \subset \mathcal{N}$, the game $G_A(\mathcal{N})$ is determined. We say that $\mathbf{AD}_\Gamma$ holds for some point class Γ if for every $A \in \Gamma$, A is determined.

Let Γ denote the projective sets. Then we also write **PD** in place of $\mathbf{AD}_\Gamma$.

Remark 5.2.20. By Theorem 5.2.4, **AD** is inconsistent with **AC**. As we shall see, **AD** localized to various point classes *will* be consistent with **AC**, and full **AD** is consistent with various weakenings of **AC**, such as **DC**.

Theorem 5.2.21. *Suppose* **AD**. *Then every point set is measurable, has the Baire property, and has the perfect set property.*

Proof. Theorem 5.2.21 is almost a corollary of Theorems 5.2.11, 5.2.14, and 5.2.18. All that is required is to show how to turn games of the form $G^*_\omega * (A)$, $G^*_2(A)$, and $G(A,\epsilon)$ into games of the form $G_A(\mathcal{N})$. However, this is essentially trivial: for any set A, the game $G^{**}_\omega(A)$ is determined only if the game $G_B(\mathcal{N})$ is, where B is the set

$$\{(n_0, m_0, n_1, m_1, \ldots) : \mathfrak{s}_{n_0}{}^\frown\mathfrak{s}_{m_0}{}^\frown\mathfrak{s}_{n_1}{}^\frown\mathfrak{s}_{m_1}{}^\frown \ldots \in A\}. \tag{5.26}$$

Likewise, the game $G(A,\epsilon)$ is determined only if $G_B(\mathcal{N})$ is, where

$$B = \{(n_0, m_0, n_1, m_1, \ldots) : (n_0, n_1, \ldots) \in {}^\omega 2 \text{ and } x \in A - \bigcup_{i<\omega} N_{m_i}; \\ \text{or, } m(N_{m_i}) \geq 2^{2(i+1)} \text{ for some } i < \omega\}. \tag{5.27}$$

Lastly, $G^*_2(A)$ is equivalent to $G_B(\mathcal{N})$, where

$$B = \{(n_0, m_0, n_1, m_1, \ldots) : (m_0, m_1, \ldots) \in \Psi(\mathcal{N}) \\ \text{and } \mathfrak{r}_{n_0}{}^\frown\langle m_0\rangle^\frown\mathfrak{r}_{n_1}{}^\frown\langle m_1\rangle^\frown \ldots \in A\} \tag{5.28}$$

where $\langle \mathfrak{r}_i : i < \omega\rangle$ is some fixed enumeration of ${}^{<\omega}2$. □

5.3 Borel Determinacy

The present section contains a proof of Borel determinacy. We follow the presentation of Martin in [Mar85], rather than [Mar75]. Our goal is to prove the following theorem.

Theorem 5.3.1. *Let T be a pruned tree in ${}^{<\omega}X$, and suppose $A \subset [T]$ is a Borel set. Then A is determined.*

We begin with a few remarks and definitions.

Remark 5.3.2. For a tree T, let $T{\restriction}_n$ denote the set $\{t \in T : |t| < n\}$. We say that a tree is *pruned to height* n if for every $s \in T$ of length less than n there exists a t extending s such that $|t| = n$. We say that T is a *pruned tree of height* n if T is pruned to height n and $|T| = n$, where $|T| = \sup\{\mathrm{OT}(t) : t \in T\}$ is the height of the tree T.

A strategy σ for Player I, then, is simply a function from $\bigcup_{n<\omega} T{\restriction}_{2n}$ to T such that $|\sigma(s)| = |s| + 1$, and a strategy τ for Player II is a map from $\bigcup_{n<\omega} T{\restriction}_{2n+1}$ to T such that $|\tau(s)| = |s| + 1$.

Rather than functions, however, strategies may be usefully thought of as *subtrees* of T which encode the next move a player should make. That is, we can think of a strategy σ for Player I as a subtree T_σ of T, where a node, if it represents a partial play p in $\mathscr{P}_{\mathrm{II}}$, is succeeded by all partial plays p' such that $p'{\restriction}_{|p|} = p$ and $p'(|p'| - 1)$ is some legal move for Player II; if the node p represents a partial play in $\mathscr{P}_{\mathrm{I}}$, then it is succeeded by the *single* node given by $\sigma(p)$. The definition for a strategy τ for Player II is exactly analogous. Since there is no risk of ambiguity, we shall not distinguish between σ and T_σ.

Then it follows that in a given infinite play p, Player I played according to σ if $p \in [\sigma]$, and likewise, that Player II played according to τ if $p \in [\tau]$.

A *partial strategy* is simply some tree of the form $T_\sigma{\restriction}_n$ or $T_\tau{\restriction}_n$ for some strategy σ for Player I or τ for Player II.

Definition 5.3.3 (Covering)**.** Let T be a pruned tree on a set X. Then, a *covering* is a triple $(\widetilde{T}, \pi, \varphi)$, where

(i) For some set $\widetilde{X}$, $\widetilde{T}$ is a tree on $\widetilde{X}$;

(ii) The map $\pi : \widetilde{T} \to T$ is a monotone map—i.e., if $t \subset s$, then $\pi(t) \subset \pi(s)$—such that $|\pi(t)| = |t|$;

(iii) The map φ takes strategies for Player I and Player II to strategies for the respective player on T in such a way that $\varphi(\widetilde{\sigma}){\restriction}_n$ depends only upon $\widetilde{\sigma}{\restriction}_0$, $\widetilde{\sigma}{\restriction}_1$, $\ldots$, $\widetilde{\sigma}{\restriction}_n$.

(iv) If $\widetilde{\sigma}$ is a strategy in $\widetilde{T}$, and p is a play of the game played according to $\varphi(\widetilde{\sigma})$—i.e., $p \in [\varphi(\widetilde{\sigma})]$—then there is some $\widetilde{x} \in [\widetilde{\sigma}]$ such that $x = \pi(\widetilde{x})$.

Note that because of the length requirement, π gives rise to a continuous map from $[\widetilde{T}]$ to $[T]$.

Further note that Condition (iii) may also be restated by saying that for any strategy σ for Player I, for each n, $\varphi(\sigma\restriction_n)$ is a pruned tree of height n that is also a partial strategy. We then can calculate

$$\varphi(\widetilde{\sigma}) = \bigcup_{n<\omega} \varphi(\sigma\restriction_n). \tag{5.29}$$

We call a covering $(\widetilde{T}, \pi, \varphi)$ a *k-covering* if $\widetilde{T}\restriction_{2k} = T\restriction_{2k}$ and $\pi\restriction_{\widetilde{T}\restriction_{2k}}$ is the identity. That is, a covering is a k-covering if $\widetilde{T}$ does not differ for the first k moves for either player from the tree T that it covers. Note that by necessity, $\varphi(\sigma\restriction_n)$ for $n < 2k$ is necessarily the identity, by Conditions (iii) and (iv) above.

Definition 5.3.4 (Unravelling)**.** Let $(\widetilde{T}, \pi, \varphi)$ cover the tree T. We say that $(\widetilde{T}, \pi, \varphi)$ *unravels* $A \subset [T]$ if $\pi^{\mathrm{PRE}}(A)$ is clopen in $[\widetilde{T}]$.

The following simple lemma demonstrates the usefulness of these definitions.

Lemma 5.3.5. *Suppose $(\widetilde{T}, \pi, \varphi)$ covers T and unravels $A \subset [T]$. Then $G_A(T)$ is determined.*

Proof. Without loss of generality, assume $\widetilde{\sigma}$ is a winning strategy for Player I in $G_{\pi^{\mathrm{PRE}}(A)}(\widetilde{T})$. We claim that $\varphi(\widetilde{\sigma})$ is a winning strategy for Player I in $G_A(T)$. For, note that if $x \in [\varphi(\widetilde{\sigma})]$, then there exists $\widetilde{x} \in [\widetilde{\sigma}]$ such that $\pi(\widetilde{x}) = x$. Since $\widetilde{\sigma}$ is a winning strategy, $\widetilde{x} \in \pi^{\mathrm{PRE}}(A)$; hence, $x \in A$. □

The notion of a k-covering is relevant on account of the following theorem.

Theorem 5.3.6. *Suppose that $A \subset [T]$ is a Borel set. Then, for each $k > 0$, there exists a k-covering of T which unravels A.*

Note that in conjunction with Lemma 5.3.5, Theorem 5.3.6 proves Theorem 5.3.1. The following two lemmata will allow us to prove Theorem 5.3.6.

Lemma 5.3.7. *Let T be a non-empty pruned tree, and let $A \subset [T]$ be closed. Then there is a covering which unravels A.*

Lemma 5.3.8 (Existence of Projective Limits)**.** *Fix $k < \omega$. For each $i < \omega$, let $(T_{i+1}, \pi_{i+1}, \varphi_{i+1})$ be a $k+n$-covering of T_n. Then, there exists a pruned tree T_ω and maps $\pi_{\omega,n}$ and $\varphi_{\omega,n}$ such that the following diagrams commute:*

$$\begin{array}{ccccccc} T_\omega & & & & & & \\ \downarrow{\scriptstyle \pi_{\omega,0}} & \searrow{\scriptstyle \pi_{\omega,1}} & & \searrow{\scriptstyle \pi_{\omega,2}} & & & \\ T_0 & \xleftarrow{\pi_1} & T_1 & \xleftarrow{\pi_2} & T_2 & \xleftarrow{\pi_3} & \cdots \end{array} \tag{5.30}$$

and

$$\begin{array}{ccccccc} \mathscr{S}(T_\omega) & & & & & & \\ \downarrow{\scriptstyle \varphi_{\omega,0}} & \searrow{\scriptstyle \varphi_{\omega,1}} & & \searrow{\scriptstyle \varphi_{\omega,2}} & & & \\ \mathscr{S}(T_0) & \xleftarrow{\varphi_1} & \mathscr{S}(T_1) & \xleftarrow{\varphi_2} & \mathscr{S}(T_2) & \xleftarrow{\varphi_3} & \cdots \end{array} \tag{5.31}$$

i.e., $\pi_{m,n} \circ \pi_{\omega,m} = \pi_{\omega,n}$ and $\varphi_{m,n} \circ \varphi_{\omega_m} = \varphi_{\omega,n}$. Moreover, $(T_\omega, \pi_{\omega,n}, \varphi_{\omega,n})$ is a $k+n$ covering of T_n.

Let us delay the proofs of Lemmata 5.3.7 and 5.3.8 for a moment, to see how they imply Theorem 5.3.6.

Proof of Theorem 5.3.6. We will prove by induction on $\alpha < \omega_1$ that there is a k-covering unraveling A for all $A \subset \mathbf{\Sigma}^0_\alpha([T])$. This suffices to prove the theorem since, if $(\widetilde{T}, \pi, \varphi)$ unravels A, then, clearly $(\widetilde{T}, \pi, \varphi)$ unravels $-A$ as well.

Now, note that by Lemma 5.3.7, the theorem holds for $\alpha = 0$. So, suppose that we have proven that Theorem 5.3.6 holds for all sets in $\mathbf{\Pi}^0_\beta[T]$ and $\mathbf{\Sigma}^0_\beta[T]$, $\beta < \alpha$. Let $A \in \mathbf{\Sigma}^0_\alpha[T]$ be arbitrary. Then, there exists a sequence of sets $\langle A_n : n < \omega\rangle$ and an increasing sequence of ordinals $\langle \beta_n : n < \omega\rangle$, all less than α, such that $A_n \in \mathbf{\Pi}^0_{\beta_n}[T]$ and $A = \bigcup_{n<\omega} A_n$.

We construct a system of coverings. Let (T_1, π_1, φ_1) be a k-covering of T unravelling X_0. Note that for each $n < \omega$, $\pi_1^{\text{Pre}}(A_n) \in \mathbf{\Pi}^0_{\beta_n}[T_1]$. Therefore, inductively define $(T_{n+1}, \pi_{n+1}, \varphi_{n+1})$ to be a $k+i$ covering of (T_n, π_n, φ_n) unwinding $\pi_{n,0}^{\text{Pre}}(A_n)$.

Then let $(T_\omega, \pi_{\omega,n}, \varphi_{\omega,n})$ be as in Lemma 5.3.8. By construction, $\pi_{\omega,0}^{\text{Pre}}(A) = \bigcup_{n<\omega} \pi_{\omega,0}^{\text{Pre}}(A_n)$ is open, since for all $n < \omega$, $\pi_{\omega,0}^{\text{Pre}}(A_n)$ is clopen.

Finally, again by Lemma 5.3.8, let $(\widetilde{T}, \widetilde{\pi}, \widetilde{\varphi})$ be a k-covering of T_ω unravelling $\pi_{\omega,0}^{\text{Pre}}(A)$. Then, $(\widetilde{T}, \pi_{\omega,0} \circ \widetilde{\pi}, \varphi_{\omega,0} \circ \widetilde{\varphi})$ is the desired k-covering of T unwinding A. □

We begin with the proof of Lemma 5.3.8.

Proof of Lemma 5.3.8. Note that because $(T_{i+1}, \pi_{i+1}, \varphi_{i+1})$ is a $(k+i)$-covering of T_i, we have the following coherence property:

$$T_n\restriction_{2(k+n)} = T_{n+1}\restriction_{2(k+n)} = T_{n+2}\restriction_{2(k+n)} = \dots . \tag{5.32}$$

Therefore, we define the tree T_ω in the following way:

$$s \in T_\omega \iff \text{There exists } n < \omega \text{ such that } |s| < 2(n+k) \text{ and } s \in T_n. \tag{5.33}$$

This definition is coherent, since if the right-hand side of Equation 5.33 holds, then $s \in T_m$ for $m > n$.

Thus, $\pi_{\omega,n}$ has a particularly simple description: if $s \in T_\omega$, then there exists some $m > n$ such that $s \in T_m$. Set $\pi_{\omega,m} = \pi_{m,n}(s)$. Again, this is well-defined by Equation 5.32.

Similarly, we define $\varphi_{\omega,n}$ in the following way: note that for all n such that $2(n+k) \geq i$, if σ_ω is a strategy in T_ω, then $\sigma_\omega\restriction_i$ is a partial strategy in T_n. Therefore, choose some $m > n$ such that $\sigma_\omega\restriction_i$ is a partial strategy in T_m, and set $\varphi_{\omega,n}(\sigma_\omega\restriction_i) = \varphi_{m,n}(\sigma_\omega\restriction_i)$. Then, to conform to Condition (iii) of Definition 5.3.3, we set

$$\varphi_{\omega,n}(\sigma_\omega) = \bigcup_{n<\omega} \varphi_{\omega,n}(\sigma_\omega\restriction_n). \tag{5.34}$$

Now, we have to check that $(T_\omega, \pi_{\omega,n}, \varphi_{\omega,n})$ actually is a $(k+n)$-covering of T_n. The only condition of Definition 5.3.3 which remains to be checked is Condition (iv). Let $x_n \in [\varphi_{\omega,n}(\sigma_\omega)]$. Then, by Condition (iv) and the fact that $\varphi_{\omega,n} = \varphi_{n+1} \circ \varphi_{\omega,n+1}$, there exists an $x_{n+1} \in [\varphi_{\omega,n+1}(\sigma_\omega)]$ such that $\pi_{n+1}(x_{n+1}) = x_n$. Likewise, there exists $x_{n+2} \in [\varphi_{\omega,n+2}(\sigma_\omega)]$ such that $\pi_{n+2}(x_{n+2}) = x_{n+1}$, and so forth. Recall that $\pi_{i+1}\restriction_{2(i+k)}$ is the identity. Therefore, if we set

$$x_\omega = \bigcup_{n<m<\omega} x_m\restriction_{2(m+k)} \tag{5.35}$$

it follows that $\pi_{\omega,n}(x_\omega) = x_n$, as desired. □

Before we proceed, we shall need the following definition.

Definition 5.3.9 (Quasistrategy)**.** Recall that, as investigated in Remark 5.3.2, a strategy (for Player I) may be thought of as a subtree σ of T at which for each node s, if $|s|$ is even, then there is exactly one node succeeding s, and if $|s|$ is odd, then every $t \in T$ such that $|t| = |s| + 1$ and $t\restriction_{|s|} = s$ is in σ.

A quasistrategy is a straightforward generalization of this concept: instead of requiring that exactly one node succeed s for s of even length, a quasistrategy (for Player I) is a subtree T' of T in which there is at least one node succeeding s for s of even length, and for every node s of odd length, *every* node t succeeding s in T is in T'. A quasistrategy for Player II is defined analogously.

Thus, recalling that nodes are equivalent to partial plays, while a strategy presents a *single* option for the player to proceed following the opposing players response, a quasistrategy presents a *choice* of options. Using the axiom of choice, a quasistrategy can, of course, be pared down to a strategy. More importantly, if a player—without loss of generality, Player I—has a *winning* strategy σ, then the tree

$$\mathcal{S}_{\mathrm{I}} = \{t \in T : t \text{ is not a losing position for Player I}\} \tag{5.36}$$

is a non-empty quasistrategy. We call this quasistrategy the *canonical quasistrategy for Player I.* If Player II has a winning strategy τ, then we can similarly define the canonical quasistrategy for Player II, $\mathcal{S}_{\mathrm{II}}$.

Proof of Lemma 5.3.7. We will construct $(\widetilde{T}, \pi, \varphi)$ by describing an auxiliary game, $G_A^+(T)$, the possible positions of which will form the tree $\widetilde{T}$.

The game is played as follows. For each of their first k turns, Player I and Player II alternate playing elements of X that are legal in $G_A(T)$. Let the partial play so formed be $p = (x_0, y_0, x_1, y_1, \ldots, x_{k-1}, y_{k-1})$. On her $(k+1)$-st turn, Player I plays (x_k, S_1), where x_k would have been a legal move in $G_A(T)$ and S_1 is a quasistrategy for the first player in the tree $T/p^\frown\langle x_k\rangle$. (Recall the definition of T/p given in Theorem 5.2.3.) Then Player II may respond in one of two ways:

(i) Player II plays (y_k, u) where y_k is a legal move in T at the position $p^\frown\langle x_k\rangle$, and $u \in S_1 - T_A$ is some sequence of even length, and where

$$T_A = \{t \in T : \text{ There exists } x \in A \text{ such that } t = x\restriction_n \text{ for some } n\}; \tag{5.37}$$

(ii) Player II plays (y_k, S_2), where again y_k is a legal move in T at the position $p^\frown\langle x_k\rangle$, and S_2 is a quasistrategy for the first player in the tree $S_1/\langle y_k\rangle$.

Thus, a partial play through Players I and II's first $k+1$ turns looks as indicated in one of the two schematics below:

$$
\begin{array}{c|c}
\multicolumn{2}{c}{G_A^+(T)\text{: Option (i)}} \\
\text{I} & \text{II} \\
\hline
x_0 & \\
 & y_0 \\
x_1 & \\
 & y_1 \\
\vdots & \\
 & \vdots \\
x_{k-1} & \\
 & y_{k-1} \\
(x_k, S_1) & \\
 & (y_k, u) \\
\vdots & \vdots
\end{array}
\qquad\qquad
\begin{array}{c|c}
\multicolumn{2}{c}{G_A^+(T)\text{: Option (ii)}} \\
\text{I} & \text{II} \\
\hline
x_0 & \\
 & y_0 \\
x_1 & \\
 & y_1 \\
\vdots & \\
 & \vdots \\
x_k & \\
 & y_k \\
(x_{k+1}, S_1) & \\
 & (y_{k+1}, S_2) \\
\vdots & \vdots
\end{array}
\tag{5.38}
$$

The rules of $G_A^+(T)$ then change depending upon whether Player II plays option (i) or option (ii). If Player II chooses Option (i), on subsequent moves the players must play elements of X that are legal at the relevant position in T and such that $\langle x_{k+1}, y_{k+1}, \ldots, x_m\rangle$ and $\langle x_{k+1}, y_{k+1}, \ldots, x_m, y_m\rangle$ are consistent with u. If Player II chooses Option (ii), then Players I and II must play their subsequent moves so that $(x_{k+1}, y_{k+1}, \ldots, x_m)$ and $(x_{k+1}, y_{k+1}, \ldots, x_m, y_m)$ are in S_2. The win condition is the same: Player I wins if $\langle x_0, y_0, x_1, y_1, \ldots\rangle \in A$, and Player II wins otherwise.

Then $\widetilde{T}$ is the tree of all legal positions in the game $G_A^+(T)$. It is easy to see that $\widetilde{T}$ is a pruned tree. Moreover, the projection map p is the obvious one given, e.g., by

$$\langle x_0, y_0, \ldots, (x_k, S_1), (y_k, u), \ldots, y_m\rangle \longmapsto \langle x_0, y_0, \ldots, x_k, y_k, \ldots, y_m\rangle \tag{5.39}$$

The particularly complicated form of the game $G_A^+(T)$ is required to ensure that $G_A^+(T)$ is a *clopen* game: Player I wins if and only if $p(2k+2) = (y_k, S_2)$ for some quasi-strategy S_2. Thus every play in which Player I wins is contained in a neighborhood of winning plays, and every play in which Player I loses is contained in a neighborhood of losing plays.

Now, it remains to construct φ. We shall do so informally, and in such a way that it is clear from the construction that Condition (iii) is satisfied. To do so, we shall break into cases.

For the remainder of the proof, p_0 will signify the partial play $\langle x_0, y_0, \ldots, x_k\rangle$ and p_1 the partial play $\langle x_0, y_0, \ldots, x_k, y_k\rangle$. Our goal is to produce strategies for Players I and II in $G_A(T)$ based on strategies in $G_A^+(T)$. To do so, each player will play $G_A(T)$ and try to *guess* some compatible partial play of $G_A^+(T)$, using their strategy in $G_A^+(T)$ to guide them.

Case 1. *The strategy $\widetilde{\sigma}$ is a strategy for Player I in $\widetilde{T}$.*

For each Player I's first k turns, she plays according to $\widetilde{\sigma}$ exactly. On her $(k+1)$-st turn, $\widetilde{\sigma}$ provides her a pair (x_k, S_1). In T, Player II responds by y_k. Then, note that since X/p_1 is closed in $[S_1/\langle y_k\rangle]$, either Player I or Player II has a winning strategy.

Subcase 1.1. *Player I has a winning strategy in the game* $G_{[S_1/\langle y_k\rangle]-X/p_1}(S_1/\langle y_k\rangle)$.

Let σ be this strategy. Then, $\varphi(\widetilde{\sigma})$ initially simply requires Player I to follow σ. Since σ is winning, after finitely many moves, some shortest position $p = \langle x_0, y_0, \ldots\ x_m, y_m\rangle$ is reached such that $p \notin T_{X/p_0}$. Then Player I realizes that Player II actually played Option (ii) in the hypothetical run of $G^+_A(T)$, and continues to play according to $\widetilde{\sigma}$ as if Player II had played (y_{k+1}, u), i.e., on her $(n+1)$-st turn, σ has Player I play

$$\widetilde{\sigma}(\langle x_0, y_0, \ldots, (x_k, S_1), (y_k, u), \ldots, x_m, y_n\rangle). \tag{5.40}$$

Then, it is clear that if a position $p = (x_0, y_0, \ldots) \in [\varphi(\widetilde{\sigma})]$ is reached in this way, then $\pi(\widetilde{p}) = p$, where

$$\widetilde{p} = \langle x_0, y_0, \ldots, (x_k, S_1), (y_k, u), \ldots\rangle. \tag{5.41}$$

Subcase 1.2. *In the game* $G_{[S_1/\langle y_{k+1}\rangle]-X/p_1}(S_1/\langle y_{k+1}\rangle)$*, Player II has a winning strategy.*

Let $\mathcal{S}_{\mathrm{II}} \subset S_1/\langle y_{k+1}\rangle$ be her canonical quasistrategy. Then, initially, $\varphi(\widetilde{\sigma})$ will instruct Player I on her $(n+1)$-st turn to play according to $\widetilde{\sigma}$ and the position in $G^+_A(T)$ given by

$$\langle x_0, y_0, \ldots, (x_k, S_1), (y_k, \mathcal{S}_{\mathrm{II}}), \ldots, x_n, y_n\rangle. \tag{5.42}$$

Now, if Player II makes a move y_m which is incompatible with this run of the game, i.e.,

$$p = \langle x_0, y_0, \ldots, (x_k, S_1), (y_k, \mathcal{S}_{\mathrm{II}}), \ldots, x_m, y_m\rangle \notin \mathcal{S}_{\mathrm{II}} \tag{5.43}$$

then, by the definition of $\mathcal{S}_{\mathrm{II}}$, Player I has a winning strategy σ starting from that position, i.e., in the game $G_{[S_1/\langle y_k,\ldots,x_m,y_m\rangle]-X/p}([S_1/\langle y_k, \ldots, x_m, y_m\rangle])$. Then, we can simply return to Subcase 1.1, assuming that Player I has played according to her winning strategy σ until the position p.

Thus, whether Player II makes a move incompatible with $\mathcal{S}_{\mathrm{II}}$ or not, if $x \in [\varphi(\widetilde{\sigma})]$, then there is some $\widetilde{x} \in [\widetilde{\sigma}]$ such that $x = \pi(\widetilde{x})$.

Case 2. *The strategy* $\widetilde{\tau}$ *is a strategy for Player II in* $\widetilde{T}$.

As before, for the first k moves, Player II plays according to $\widetilde{\tau}$. Then, Player I plays x_k. Again let $p_0 = (x_0, y_0, \ldots, x_k)$. Let

$$\mathcal{W} = \{S : S \text{ is a quasistrategy for the first player in } T/p_0\} \tag{5.44}$$

and let

$$\mathcal{U} = \{\langle y_k\rangle^\frown u : |u| \text{ is even and there is an } S \in \mathcal{W} \text{ such that } \widetilde{\tau} \text{ asks Player II to play } (y_k, u) \text{ in response to } (x_k, S)\}. \tag{5.45}$$

Then observe that the set

$$U = \{x \in T/p_0 : \text{There exists } \langle y_k\rangle^\frown u \in \mathcal{U} \text{ such that } x \text{ extends } \langle y_k\rangle^\frown u\} \tag{5.46}$$

is trivially open in $[T/p_0]$. Consider the following game:

$$
G^{\circ}_A(T)
\qquad
\begin{array}{c|c}
\text{I} & \text{II} \\
\hline
 & y_k \\
x_{k+1} & \\
 & y_{k+1} \\
x_{k+2} & \\
 & y_{k+2} \\
x_{k+3} & \\
\vdots & \vdots
\end{array}
\tag{5.47}
$$

Here, Player II (playing first in this game) wins if the infinite play p is in U, and Player I wins otherwise. Since U is open, either Player I or Player II has a winning strategy.

Subcase 2.1. *Player II has a winning strategy τ in $G^{\circ}_A(T)$.*

Then Player II continues playing in $G^+_A(T)$ according to τ until, necessarily at some finite stage, a sequence $\langle y_k, x_{k+1}, \dots, x_m, y_m\rangle \in \mathcal{U}$ is reached. Let S witness that $\langle y_k\rangle^\frown u \in \mathcal{U}$, where $u = \langle x_{k+1}, \dots, x_m, y_m\rangle$. Then, Player II simply follows $\widetilde{\tau}$ as if Player I had played (x_k, S) on her $(k+1)$-st turn, and Player II had responded by (y_k, u).

Subcase 2.2. *Player I has a winning strategy in $G^{\circ}_A(T)$.*

Let $\mathcal{S}_{\text{I}}$ be her canonical winning quasistrategy. Then $\widetilde{\tau}$ cannot ask Player II to respond to $(x_k, \mathcal{S}_{\text{I}})$ with a move of the form (y_k, u), since then $\langle x_k\rangle^\frown u \in \mathcal{U}$, and by the rules of $\widetilde{T}$, $\langle x_k\rangle^\frown u$ would therefore be in $\mathcal{S}_{\text{I}}$. But $\langle x_k\rangle^\frown u$ is a losing position for Player I in $G^{\circ}_A(T)$.

Then, $\widetilde{\tau}$ asks Player II to play (y_k, S_2) for some quasistrategy S_2. Player II continues playing in $G_A(T)$ as if Player I had played $(x_k, \mathcal{S}_{\text{I}})$ and Player II had played (y_k, S_2) on their respective $(k+1)$-st turns. If Player I eventually makes a move x_m such that the sequence $(x_{k+1}, y_{k+1}, \dots, x_m) \notin S_2$—i.e., the position in the hypothetical run of $G^+_A(T)$ is no longer legal—then, since S_2 is a quasistrategy, it follows that $(y_k, x_{k+1}, y_{k+1}, \dots, x_m) \notin \mathcal{S}_{\text{I}}$, as quasistrategies do not restrict the other players possible moves. Therefore, by the definition of $\mathcal{S}_{\text{I}}$, Player II has a winning position strategy σ in the game $G^{\circ}_A(T)$ beginning at the position

$$p = (y_k, x_{k+1}, y_{k+1}, \dots, x_m) \tag{5.48}$$

and so we can return to Subcase 2.1

In either case, it is clear that if $x \in [\tau]$, then there is $\widetilde{x} \in [\widetilde{t}]$ such that $\pi(\widetilde{x}) = x$. □

This completes the proof of Theorem 5.3.1.

Remark 5.3.10. As the construction of the game $G^+_A(T)$ indicates, Borel determinacy requires one to lift the games on $\mathcal{N}$ to games on trees of successively larger and larger cardinality. This was an early indication that determinacy hypothesis required more power to prove than, say, is contained $V_{\omega+\omega}$, a set which suffices as a framework for almost all of normal mathematics.

Even though Martin had published the proof of Borel determinacy in 1975 in [Mar75], Friedman had already proven that Borel determinacy cannot be proven in Zermelo set theory, i.e., without the existence of V_{ω_1} [Fri71]. The developments detailed in the next section indicate how quickly this connection between the height of V and determinacy grew.

5.4 Large Cardinals

Two important advances connecting large cardinals with determinacy hypotheses are considered in this section. The first is the proof of $\mathbf{\Pi}^1_1$-determinacy from the existence of a measurable cardinal, given by Martin in [Mar70]. The second is Martin's proof of $\mathbf{\Pi}^1_2$-determinacy from the existence of an iterable cardinal, first published in [Mar80].

5.4.1 Measurable cardinals and $\mathbf{\Pi}^1_1$-determinacy

Definition 5.4.2 (Measurable Cardinal)**.** Recall that a cardinal κ is *measurable* if there exists a non-trivial κ-complete ultrafilter U on $\mathcal{P}(\kappa)$. An ultrafilter is *nontrivial* if $\{\alpha\} \notin U$ for any $\alpha \in \kappa$. (See Chapter 10 in [Jec03].)

Definition 5.4.3. A κ-complete ultrafilter U on $\mathcal{P}(\kappa)$ is said to be *normal* if it is closed under *diagonal intersections*, i.e., for a given sequence of measure one sets $\langle Y_\alpha : \alpha < \kappa \rangle$, $\triangle_{\alpha<\kappa} Y_\alpha \in U$, where

$$\triangle_{\alpha<\kappa} Y_\alpha = \{\alpha < \kappa : \alpha \in \bigcap_{\beta<\alpha} Y_\beta\}. \tag{5.49}$$

Normal ultrafilters can be equivalently characterized by the property that if $g : \kappa \to \kappa$ is such that $\{\alpha < \kappa : g(\alpha) < \alpha\} \in U$, then there is some $X \in U$ such that $g{\restriction}_X$ is constant.

Remark 5.4.4. It is not hard to see that a measurable cardinal κ possesses a diagonal ultrafilter: if ${}^\kappa\kappa$ is ordered according to

$$f <_U g \iff \{\alpha \in \kappa : f(\alpha) < g(\alpha)\} \in U \tag{5.50}$$

then it is not hard to see that $<_U$ is a well-ordering, modulo the equivalence

$$f \sim g \iff \{\alpha < \kappa : f(\alpha) = g(\alpha)\} \in U. \tag{5.51}$$

Therefore, if f_0 is some $<_U$-minimal element such that for all $\beta < \kappa$, the set $f^{\text{PRE}}(\beta) \notin U$, then a routine calculation verifies that the ultrafilter $f_0^{\text{PRE}}(U) = \{X \subset \kappa : f_0^{\text{PRE}}(X) \in U\}$ is normal.

Definition 5.4.5 (Homogeneity)**.** Let κ be a cardinal and $[\kappa]^m$ denote the subsets of κ of size m. If $f : [\kappa]^m \to \gamma$ is any map, and $\gamma < \kappa$ is some cardinal, we say that $H \subset \kappa$ is *homoegeous* for f if $f{\restriction}_{[H]^m}$ takes only a single value.

That is, if f partitions $[\kappa]^m$ into γ-many pieces, then if H is a homogenous subset of κ, then every size m subset of H is mapped to the *same* piece.

Similarly, we say that H is homomgeneous for $f : [\kappa]^{<\omega} \to \gamma$, where $[\kappa]^{<\omega} = \bigcup_{m<\omega} [\kappa]^m$, if H is homogeneous for the restriction of f to $[\kappa]^m$ for each $m < \omega$.

Then, we have the following lemma.

Lemma 5.4.6. *Suppose κ is a measurable cardinal as witnessed by the ultrafilter U, and $f : [\kappa]^{<\omega} \to \gamma$ is as above. Then there is a set H homogeneous for f such that $H \in U$.*

Proof. Notice that since U—which we shall without loss of generality assume is normal—is κ-complete, it suffices to show that there exists some homogeneous set for $f\restriction_{[\kappa]^m}$ for each m. Therefore, we proceed by induction.

In the case $m = 1$, note that by κ-completeness, since $\gamma < \kappa$, f is necessarily constant on some set $H \in U$. Next, suppose the result has been established for m. Then, for each $s \in [\kappa]^m$, define

$$f_s : \kappa \to \kappa \qquad f_s(\beta) = \begin{cases} f(s \cup \{\beta\}) & \beta > \max(s) \\ 0 & \beta \leq \max(s) \end{cases} \tag{5.52}$$

By κ-completeness, there is some δ_s and a set $Y_s \in U$ such that $f_s\restriction_{Y_s}$ takes the constant value δ_s. By the inductive hypothesis applied to the map $s \mapsto \delta_s$, there is therefore some set $Y \in U$ and a fixed δ such that $\delta_s = \delta$ for all $s \in [Y]^m$. Let $Y_\alpha = \bigcap_{\max(s)<\alpha} Y_s$. Then $Y_\alpha \in U$, and, by normality, the set $H = Y \cap \triangle_{\alpha<\kappa} Y_\alpha$ is in U also. Let $t \in [H]^{m+1}$; then, we can split t into a subset s and an ordinal β such that $\beta > \max(s)$; hence, since $\beta \in H$, $\beta \in \triangle_{\alpha<\kappa} Y_\alpha$, so $\beta \in \bigcap_{\alpha<\beta} Y_\alpha$. Since $\max(s) < \beta$, it follows that $\beta \in Y_s$, so $f(t) = f_s(\beta) = \delta$. Therefore H is the desired homogeneous set. □

Remark 5.4.7. Recall that the analytic sets, among several other equivalent definitions, can be defined using the $\mathcal{A}$-operation; namely, for any analytic set X, there is a collection of closed sets $\{V_s : s \in {}^{<\omega}\omega\}$ such that

$$x \in X \iff x \in \mathcal{A}(\{V_s\}) \iff (\exists f : \omega \to \omega)(\forall n < \omega)(x \in V_{f\restriction_n}). \tag{5.53}$$

(See Lemmata 11.6 and 11.7 in [Jec03].) Put slightly differently, functions $f : \omega \to \omega$ correspond to branches of the tree ${}^{<\omega}\omega$. Each node s of the branch correponds to an intersection of closed sets $V_{s\restriction_0} \cap V_{s\restriction_1} \cap \ldots \cap V_{s\restriction_{|s|}}$. If $x \in -X$, then no matter *which* branch $f : \omega \to \omega$ we choose, at some node along that branch, x fails to be in the intersection corresponding to that node. Therefore, one can consider all of the nodes of ${}^{<\omega}\omega$ for which x *is* in the corresponding intersection, and consequently see that this tree is well founded, i.e., it has no infinite increasing chain of nodes.

In this way, the co-analytic sets are seen to correspond to trees on $\omega \times \omega$—that is, a collection of pairs of finite sequences in ${}^{<\omega}\omega$ of the same length which, if it contains (s, t), contains $s\restriction_n, t\restriction_n$ for all $n < |s| = |t|$. If T corresponds to the co-analytic set X, then we have

$$x \in X \iff T_x \text{ is well-founded} \tag{5.54}$$

where $T_x = \{s \in {}^{<\omega}\omega : (s, x\restriction_{|s|}) \in T\}$.

(See Section 32 of [Kec95].)

Theorem 5.4.8. *Suppose there exists a measurable cardinal κ. Then all $\mathbf{\Pi}^1_1$ sets are determined.*

Proof. Let A be a $\mathbf{\Pi}^1_1$ set, and let T be its corresponding tree. Note that for any tree $T \subset {}^{<\omega}\omega$, S is well-founded if and only if the *Kleene-Brouwer ordering* restricted to S is a well-ordering, where for s and t in ${}^{<\omega}\omega$,

$$\begin{aligned} s <_{\mathrm{KB}} t \iff t \subset s \text{ or, if } m \text{ is the greatest integer} \\ \text{such that } s{\restriction}_m = t{\restriction}_m, \text{ then } s(m+1) < t(m+1). \end{aligned} \tag{5.55}$$

Note that if S is well-founded, simply by looking at the order type of each node, then the Kleene-Brouwer ordering provides an embedding of $(S, <_{\mathrm{KB}})$ into $(\kappa, \in)$. (In fact, the embedding can be made to be into $(\omega_1, \in)$ purely by cardinality considerations.) Moreover, clearly such an embedding can only exist if S is well-founded.

Then, we define a game $G^\star_A(\mathcal{N})$ on $\omega \times \kappa$ as indicated in the schematic below:

$$\begin{array}{cc} G_A(\mathcal{N}) & G^\star_A(\mathcal{N}) \\ \begin{array}{c|c} \text{I} & \text{II} \\ \hline n_0 & \\ & m_0 \\ n_1 & \\ & m_1 \\ n_2 & \\ & m_2 \\ \vdots & \vdots \end{array} & \begin{array}{c|c} \text{I} & \text{II} \\ \hline (\beta_0, n_0) & \\ & m_0 \\ (\beta_1, n_1) & \\ & m_1 \\ (\beta_2, n_2) & \\ & m_2 \\ \vdots & \vdots \end{array} \end{array} \tag{5.56}$$

The rules of the game $G^\star_A(\mathcal{N})$ are similar to the rules of $G_A(\mathcal{N})$: Player I, on her turn, plays an integer and an ordinal, and Player II plays integers. Player I wins if the resulting real number they define, $x = (n_0, m_0, n_1, m_1, \ldots)$, is in A *and* the ordinals $\langle \beta_i : i < \omega \rangle$ witness that T_x is well-founded in the following way: recalling the fixed enumeration of finite sequences $\mathfrak{s}_0$, $\mathfrak{s}_1$, $\mathfrak{s}_2$, ..., if $\mathfrak{s}_i, \mathfrak{s}_j \in T_x$ and $\mathfrak{s}_i <_{\mathrm{KB}} \mathfrak{s}_j$, then $\beta_i < \beta_j$; if $\mathfrak{s}_i \notin T_x$, then $\beta_i = 0$. Player II wins otherwise.

We make the following important observation: if $x{\restriction}_n = x'{\restriction}_n$, then $T_x{\restriction}_n = T_{x'}{\restriction}_n$. From this, we can deduce that $G^\star_A(\mathcal{N})$ is a *closed* game, for, if Player I loses on some play p, then the sequence $\langle \beta_i : i < \omega \rangle$ fails to witness the desired embedding, and hence does so at some finite stage, i.e., $\mathfrak{s}_i <_{\mathrm{KB}} \mathfrak{s}_j$, but $\beta_i \geq \beta_j$. Then, by the observation above, any play which agrees up to Player I's $(\max\{i, j\})$-th turn will also be a losing play for Player I.

Therefore, $G^\star_A(\mathcal{N})$ is determined. Clearly, if Player I has a winning strategy σ in $G^\star_A(\mathcal{N})$, then she has a winning strategy in $G_A(\mathcal{N})$: she simply does not reveal what ordinals σ asks her to play.

Now, suppose Player II has a winning strategy $\tau^\star$ in $G^\star_A(\mathcal{N})$. We seek to construct a winning strategy τ for her in $G_A(\mathcal{N})$. Toward, that end, for each sequence $s \in {}^{<\omega}\omega$ of even length $2m$, we shall associate a function $f_s : [\kappa]^m \to \omega$. To construct f_s, we must first note that for all $a \in [\kappa]^m$, there is trivially a unique map $g^{a,s} : m \to a \cup \{0\}$ such that

1. For all $\mathfrak{s}_i \in T_s = \{t \in T : t \subset s \text{ or } s \subset t\}$, $\beta^{s,a}_i \in a$;
2. For $\mathfrak{s}_i \notin T_s$, $\beta^{s,a}_i = 0$;

3. For $\mathfrak{s}_i, \mathfrak{s}_j \in T_s$, if $\mathfrak{s}_i <_{\mathrm{KB}} \mathfrak{s}_j$, then $\beta_i^{s,a} < \beta_j^{s,a}$;

4. Only the n smallest elements of a are in the range of $g^{a,s}$, where n equals $|\{\mathfrak{s}_0, \mathfrak{s}_1, \ldots\} \cap T_s|$.

Then, we set

$$f_s(a) = \tau\big(\langle g^{a,s}(0), s(0)\rangle, s(1), \langle g^{a,s}(1), s(2)\rangle, \ldots, \langle g^{a,s}(m), s(2m)\rangle\big). \tag{5.57}$$

By Lemma 5.4.6, there is some set H in U which is homogeneous for f_s. Then, in the game $G_A(\mathcal{N})$, if the current partial play is given by $s \in {}^{2m}\omega$, τ requires Player II to play $f_s(a)$ for $a \in [H]^m$.

Now, we claim that τ is a winning strategy for Player II in $G_A(\mathcal{N})$. Suppose not; then let $x = (n_0, m_0, \ldots)$ represent an infinite play in which Player II played according to the strategy just described. If Player II loses, then T_x is well-founded. Let H_m be the homogenous set associated to $f_{x\restriction_{2m}}$, as described above. Then, since $H_\omega = \bigcap_{m<\omega} H_m \in U$, and, moreover, $|H_\omega| > \omega_1$, it follows that there is an embedding of $(T_x, <_{\mathrm{KB}})$ into $(H_\omega, \in)$. Let $\langle \beta_i : i < \omega\rangle$ be constructed so that if $\mathfrak{s}_i \in T_x$, $\beta_i \in H$, if $\mathfrak{s}_i \notin T_x$, then $\beta_i = 0$, and if $\mathfrak{s}_i <_{\mathrm{KB}} \mathfrak{s}_j$ for $\mathfrak{s}_i, \mathfrak{s}_j \in T_x$, then $\beta_i < \beta_j$. Then, the sequence

$$(\langle n_0, \beta_0\rangle, m_0, \langle n_1, \beta_1\rangle, m_1, \ldots) \tag{5.58}$$

represents a play of the game where Player II played according to $\tau^\star$ and lost. But $\tau^\star$ was assumed a winning strategy. Therefore, τ is a winning strategy for Player II in $G_A(\mathcal{N})$ as well. □

5.4.9 Iterable cardinals and $\mathbf{\Pi}^1_2$-determinacy

Martin soon extended his results to the next highest class: the $\mathbf{\Pi}^1_2$ sets. His proof, of which we shall give a relatively complete sketch in this section, provided the inspiration for the proof of **PD** given in Chapter 6. It also provided new evidence in support of the more and more widely held belief that determinacy, measured against the large cardinal hierarchy, was an extremely strong hypothesis. While measurable cardinals are the smallest of the *large* cardinals—i.e., those cardinals whose existence is inconsistent with $V = L$—the very next step seemed to require climbing essentially to the *top* of the known large cardinal hierarchy.

Since many of the technical details required to prove Theorem 5.4.14 parallel those developed in Chapter 6, we omit certain proofs.

Definition 5.4.10. Let $j : V_\lambda \to V_\lambda$ for some cardinal λ, where $\kappa_0 = \mathrm{CRT}(j)$, $\kappa_{n+1} = j(\kappa_n)$, and $\lim_{n\to\omega} \kappa_n = \lambda$. Then, we say that j is *iterable* if the direct limit of the sequence

$$V_\lambda \xrightarrow{j_0} V_\lambda \xrightarrow{j_1} V_\lambda \xrightarrow{j_2} \cdots \tag{5.59}$$

is well-founded, where $j_0 = j$ and $j_{n+1} = j_n(j_n)$.

Definition 5.4.11 (β^*-Embeddings)**.** A *β^*-embedding* is an elementary embedding $k : V_{\alpha+\beta} \to V_{\alpha'+\beta}$, where $\alpha = \mathrm{CRT}(k)$. For any β^*-embedding k, let $\nu(k) = \alpha$ and $\nu'(k) = \alpha'$, so that $k : V_{\nu(k)+\beta} \to V_{\nu'(k)+\beta}$. We denote the set of β^*-embeddings by $\mathcal{I}^*_\beta$.

For any $\gamma+1 < \beta$, there is a *canonical 0-1 measure associated j on γ^*-embeddings* given by

$$\mu_k^\gamma(X) = 1 \iff j\restriction_{\nu(k)+\gamma} \in k(X). \tag{5.60}$$

(Note that this is well-defined, since the set of γ^*-embeddings is an element of $V_{\nu(k)+\beta}$.)

Lemma 5.4.12. *Let k be a β^*-embedding. The measure μ_k^β is $\nu(k)$-complete and concentrates on γ^* embeddings h for $\gamma+1<\beta$ such that $\nu(h) < \nu(k)$ and $\nu'(h) = \nu'(k)$. Moreover,*

(i) If $\gamma+1 < \beta' < \beta$, then

$$\mu_k^\gamma = \mu^\gamma_{k\restriction_{V_{\nu(k)+\beta'}}} \tag{5.61}$$

(ii) If $\gamma_1 < \gamma_2$, and $\gamma_2 + 1 < \beta$, then

$$\mu_k^{\gamma_2}\{z : z\restriction_{V_{\nu(z)+\gamma_1}} \in X\} = \mu_k^{\gamma_1}(X) \tag{5.62}$$

for $X \in V_{\nu(k)+\gamma_1}$.

All of the assertions of Lemma 5.4.12 reduce to a routine calculation, so we omit the proof.

Lemma 5.4.13 ($\mathbf{\Pi}^1_2$ Normal Form)**.** *Suppose j is iterable. Let A be a $\mathbf{\Pi}^1_2$ subset of $\mathcal{N}$. Then, there is an associated function $\rho_A : {}^{<\omega}\omega \times {}^{<\omega}\omega \to \omega$ such that for any $x \in \mathcal{N}$, $x \in A$ if and only if there is a function $H : {}^{<\omega}\omega \to \lambda$, depending on x, such that:*

1. *For any $t \in {}^{<\omega}\omega$, if $|t| = 1$, then $H(t) < \kappa_0$;*
2. *If $t'\restriction_{|t|} = t$, $|t'| = |t|+1$, then $H(t') < j_{\rho(x\restriction_{|t'|},t')}(H(t))$.*

Proof. See Lemmata 4.1 and 4.2 in [Mar80]. □

Theorem 5.4.14. *All $\mathbf{\Pi}^1_2$ subsets of $\mathcal{N}$ are determined.*

To prove this, we shall need the following notion.

Definition 5.4.15 (Iterated Product Measures)**.** Let $\langle Y_i : i < \omega\rangle$ and $\langle \mathscr{B}_i : i < \omega\rangle$ be given, where β_n is a σ-algebra of sets on y_n, along with a measure μ_0 on Y_0. Suppose that M is a function mapping $Y_0 \times \cdots \times Y_i$ to 0-1 measures on Y_{n+1} for all $n < \omega$. Then we say that M gives rise to the *iterated product measure* μ_M, where μ_M measures subsets of $Y_1 \times \cdots \times Y_n$ for all $n < \omega$, in the following way: $\mu_M(S) = \mu_0(S)$ for $S \subset Y_1$ measurable; then, supposing we have defined μ_M for sets of n-tuples, we can define μ_M for $S \subset Y_0 \times \cdots \times Y_n$ by

$$\mu_M(S) = \mu_M\{(y_0,\ldots,y_{n-1}) : M(y_{n-1})[\{y_n \in Y_{n+1} : (y_0,\ldots,y_n) \in S\}] = 1\}. \tag{5.63}$$

(Note that we can only carry out the construction if Equation 5.63 is coherent; see Remark 5.4.16 below.) We call

$$\{y_n \in Y_{n+1} : (y_0,\ldots,y_n) \in S\} \tag{5.64}$$

the fiber of S over $\vec{y} = (y_0, \ldots, y_{n-1})$, and denote it $S_{\vec{y}}$. Then, put differently, the iterated product measure is recursively defined by setting $\mu_M(S)$ to be the measure of the set of $\vec{y} \in \pi_i(S)$ over which the fiber $S_{\vec{y}}$ has $M(\vec{y})$-measure one.

Note that given a finite sequence $\langle Y_i : i < n \rangle$ otherwise satisfying the same conditions, one can carry out exactly the same construction, giving what we shall also call an iterated product measure on k-tuples for $0 < k \leq n$.

Remark 5.4.16. Note that an iterated product measure may not be defined in general because the set of points with measure one fibers, i.e., the set on the right hand side of Equation 5.63, may fail to be measurable. We also require that for any μ_M-measurable set S in $\mathscr{B}_i$ that π_{i-1}"S is also μ_M-measurable, where π_{i-1} is the canonical projection from $Y_0 \times \cdots \times Y_i$ to $Y_0 \times \cdots \times Y_{i-1}$. Since the measures that concern us in the course of the proof of Theorem 5.4.14 are all generated by σ-complete ultrafilters, we can safely put this issue aside.

Lemma 5.4.17. *Let μ_M be an iterated product measure, and let $\langle S_i : i < \omega \rangle$ be a sequence of measure one sets, where $S_i \in \mathscr{B}_i$. Then, there exists a sequence of $\langle x_i : i < \omega \rangle$, where $x_i \in Y_i$ and $\pi_i(x_{i+1}) = x_i$.*

Remark 5.4.18. The sequence of points $\langle x_i : i < \omega \rangle$ is called a *thread*. See Definition 6.4.1 in Chapter 6.

Proof. The proof is essentially identical to the proof of Lemma 6.6.2 in Chapter 6: one merely reduces to subsets $\langle S'_i : i < \omega \rangle$ where every point $\vec{x} \in Y_0 \times \cdots \times Y_{i-1}$ lies below a fiber of full measure in S'_i, π_i"S'_{i+1}, $\pi_{i+2,i}$"S_{i+2}, and so forth, and then chooses $x_i \in Y_i$ arbitrarily in each fiber. ☐

Proof of Theorem 5.4.14. Let A be a $\mathbf{\Pi}^1_2$ subset of $\mathcal{N}$. We note the following auxiliary game, $G^{\#}_A(\mathcal{N})$, which is played as follows:

$$
\begin{array}{c|c}
\multicolumn{2}{c}{G_A(\mathcal{N})} \\
\text{I} & \text{II} \\
\hline
n_0 & \\
 & m_0 \\
n_1 & \\
 & m_1 \\
n_2 & \\
 & m_2 \\
\vdots & \vdots
\end{array}
\qquad
\begin{array}{c|c}
\multicolumn{2}{c}{G^{\#}_A(\mathcal{N})} \\
\text{I} & \text{II} \\
\hline
(\alpha_0, n_0) & \\
 & m_0 \\
(\alpha_1, n_1) & \\
 & m_1 \\
(\alpha_2, n_2) & \\
 & m_2 \\
\vdots & \vdots
\end{array}
\tag{5.65}
$$

where the rules are essentially the same as in $G_A(\mathcal{N})$, except that Player I is required to play on her n-th turn the ordinal α_n. Player I wins if the resulting real $x = (n_0, m_0, \ldots)$ is in A, and if the function H given by $\mathfrak{s}_i \mapsto \alpha_i$ satisfies the requirements of Lemma 5.4.13.

In essentially the same way as before, we see that $G^{\#}_A(\mathcal{N})$ is an open game on $\omega \times \lambda$: if Player I loses, then she must have failed to correctly play an ordinal at some finite turn. Every play which agrees with the losing play up to that turn will likewise be losing.

Now, it is easy to see that if Player I has a winning strategy in $G^{\#}_A(\mathcal{N})$, then she has a winning strategy in $G_A(\mathcal{N})$.

Now, suppose Player II has a winning strategy τ in $G^{\#}_A(\mathcal{N})$. We describe a strategy for Player I in $G_A(\mathcal{N})$. To do so, we build an iterated product measure. Suppose that p is a partial play of $G_A(\mathcal{N})$ consisting of the players' first n turns, with Player II to play. We know by induction that there are ordinals β_0, β_1, ..., β_{n-1} such that if p is augmented to a position $p^{\#}$ in $G^{\#}_A(\mathcal{N})$ by the addition of β_i to Player I's i-th move, then the position $p^{\#}$ is not losing for Player II.

Then, to construct an iterated product measure, it suffices to give a description of the function M. Given the i-tuple $k_0, \ldots, k_{i-1}$, if $|\mathfrak{s}_i| = 1$, then we set

$$M(k_0, \ldots, k_{i-1}) = \mu^{\beta_i}_{j_1 \restriction V_{\kappa_1+\beta_i+2}}. \tag{5.66}$$

Otherwise, let the one term truncation of $\mathfrak{s}_i$ be $\mathfrak{s}_{i'}$—i.e., $\mathfrak{s}_i \restriction_{|\mathfrak{s}_i|-1} = \mathfrak{s}_{i'}$—and set

$$M(k_0, \ldots, k_{i-1}) = \mu^{\beta_{i'}}_{j_m(k_{i'})} \tag{5.67}$$

where $m = \rho(p\restriction_{|\mathfrak{s}_i|}, \mathfrak{s}_i)$.

Then, for a measure one set of $\mathcal{I}_{\beta_0} \times \mathcal{I}_{\beta_{n-1}}$ $\tau(P(k_0, \ldots, k_{n-1}))$ takes a constant value, where $P(k_0, \ldots, k_{n-1})$ is the position in $G^{\#}_A(\mathcal{N})$ obtained by augmenting p by setting $\beta_i = \nu(k_i)$. Note that the move obtained for Player II is independent of our choice of $\langle \beta_i : i < n \rangle$ by Lemma 5.4.12.

Then, we claim that this gives a winning strategy for Player II. For, if it did not, then let $x = (n_0, m_0, \ldots)$ be some infinite play such that Player II played according to this strategy and lost. Let H witness that $x \in A$; then, letting $\beta_i = 2H(\mathfrak{s}_i)$, and using the β_i to compute an iterated product measure, we could, by Lemmata 5.4.12 and 5.4.17 find a sequence of β_i^*-embeddings $(k_0, k_1, \ldots)$ such that setting $\alpha_i = \nu(k_i)$, the mapping $\mathfrak{s}_i \mapsto \alpha_i$ satisfies the requirements of H. Therefore, x can be extended to a play of the game where Player I plays (n_i, α_i) and Player II responds by m_i. (Recall that $(k_0, k_1, \ldots)$ are chosen from the set of measure one on which τ requires Player II to respond by m_i on her i-th turn.) Therefore Player II loses in $G^{\#}_A(\mathcal{N})$ while playing according to τ, which is impossible. Thus, Player II has a winning strategy in $G_A(\mathcal{N})$, as desired. □

Chapter 6

$\mathbf{I}_0$ implies PD

6.1 Introduction

This chapter is devoted to the proof of projective determinacy from the large cardinal axiom $\mathbf{I}_0$ originally discovered by Hugh Woodin in the fall of 1983. As outlined in Chapter 5, the evidence accumulating in the 1970's and early 1980's pointed to projective determinacy's being a very strong hypothesis relative to the large cardinal hierarchy. Following Kenneth Kunen's proof of the inconsistency in **ZFC** of a Reinhardt cardinal, or, what amounts to the same thing, the existence of an elementary embedding $j : V \to V$, the large cardinal axioms $\mathbf{I}_1$, $\mathbf{I}_2$, and $\mathbf{I}_3$ were originally isolated as moderate weakenings of the earlier notion to which the methods of Kunen's proof could not be applied. In their 1978 *Strong Axioms of Infinity and Elementary Embeddings*, Solovay, Reinhardt, and Kanamori wrote that "It seems likely that I1, I2 and I3 are all inconsistent since they appear to differ from the proposition proved inconsistent by Kunen only in an inessential technical way" [SRK78, p. 109]. Nevertheless, Martin's isolation of a hypothesis strictly between $\mathbf{I}_2$ and $\mathbf{I}_3$—the iterability hypothesis of Definition 5.4.10—and subsequent proof of $\mathbf{\Pi}^1_2$-determinacy from this hypothesis lent further credence to their consistency.

It was in this context that Hugh Woodin isolated in 1983 the new, stronger large cardinal concept, $\mathbf{I}_0$, and used this new axiom to first prove all projective sets are determined and then shortly later, that all the sets in $L(\mathbb{R})$ are determined. The strength of this notion conformed to the general sense that Martin's proof was, in some sense, the "right" proof, and that the large cardinal hypotheses necessary to prove a certain amount of determinacy were equivalent to the large cardinal hypotheses necessary to prove roughly the same amount of measurability. However, soon after the discovery of his proof, Foreman, Magidor, and Shelah independently proved the consistency of Martin's Maximum relative to the existence of a supercompact cardinal [Kan12, 460]. Woodin, on hearing of the Foreman-Magdor-Shelah results, realized that the existence of a supercompact cardinal implies the all the projective sets are Lebesgue measurable; Woodin and Shelah then improved this to all sets in $L(\mathbb{R})$.

The firm indication was that the large cardinal concepts which had been isolated to prove determinacy were far stronger than they needed to be. The result regarding projective measurability seemed to close the door on the possibility that $\mathbf{I}_0$ was even close to the optimal hypothesis for proving **PD**, and Woodin's proof was never published.

While the second thrust of the determinacy program was ultimately successful in determining the exact consistency strength of **PD** relative to large cardinal hypotheses, recently, there has been renewed interest in the strong hypotheses that characterized its beginning. Such hypotheses make possible generalizations of descriptive set theory from $V_{\omega+1}$ to $V_{\lambda+1}$ for a large cardinal λ, and have demonstrated unexpected ties to certain elementary algebraic problems. The possibility of deep structural connections between $\mathbf{I}_0$ and Ultimate L has lent further momentum to the renaissance. Recent works of Laver, (e.g., [Lav01], [Lav97], and [Lav95]) Woodin (e.g., [Woo11a] and [Woo11c]), and Cramer (e.g., [Cra15]) speak to relevance of $\mathbf{I}_0$ and other strong axioms of infinity to contemporary research in set theory.

The majority of the present chapter is a relatively self-contained presentation of Woodin's original proof of projective determinacy from $\mathbf{I}_0$, written in part based on [Ste84]. Nevertheless, the proofs of some results are sketched only in favor of concentrating upon what is substantially different in this proof from other proofs of **PD** in the literature. Section 6.8 provides a sketch provides a sketch of how to extend the proof of projective determinacy to a proof that all the sets in $L(\mathbb{R})$ are determined.

6.2 Strong Axioms of Infinity

The various large cardinal notions with which we shall be concerned are formulated in terms of an elementary embedding j. The notion of an elementary embedding is not formalizable in $\mathcal{L}_\epsilon$ by Tarski's Theorem, but in general one can augment $\mathcal{L}_\epsilon$ with the function symbol j along with a schema of axioms, one for each $\varphi \in \textsc{Form}$, expressing j's elementariness.

We first quote the following theorem of Kunen.

Theorem 6.2.1 (Kunen). *If $j : V_\delta \to V_\delta$, then $\delta = \lambda$ or $\delta = \lambda + 1$, where $\kappa_n = \textsc{Crt}(j_n)$, and $\lambda = \lim_{n\to\omega} \kappa_n$.*

Proof. See [Kan94, p. 322]. □

The strong large cardinal axioms that will concern us are the following:

($\mathbf{I}_0$): We say that $\mathbf{I}_0(\lambda, j)$ holds if $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ is an elementary embedding.

($\mathbf{I}_1$): We say that $\mathbf{I}_1(\lambda, j)$ holds if $j : V_{\lambda+1} \to V_{\lambda+1}$ is an elementary embedding.

($\mathbf{I}_2$): If $j : V \to M$ is an elementary embedding for some transitive model M such that $V_\lambda \subset M$, we say that $\mathbf{I}_2(\lambda, j)$ holds.

($\mathbf{I}_E$): The large cardinal axiom $\mathbf{I}_E(\lambda, j)$ holds if there is an elementary embedding $j : V_\lambda \to V_\lambda$ such that the direct limit of the sequence of embeddings

$$V_\lambda \xrightarrow{j} V_\lambda \xrightarrow{j_1} V_\lambda \xrightarrow{j_2} \cdots$$

is well-founded, in the sense that for any $\alpha < \lambda$, $j_{0,\omega}(\alpha)$ is well-founded.[1]

[1]Let $\mathcal{M}$ denote the direct limit. Note that it is always true that $\mathcal{M} \models$ "$j_{0,\omega}(\alpha)$ is well-founded"; however, there may be an infinite descending sequence of ordinals all $\epsilon_{\mathcal{M}}$-less than $j_{0,\omega}(\alpha)$ which exists in V but not in $\mathcal{M}$.

($\mathbf{I}_3$): We say that $\mathbf{I}_3(\lambda, j)$ holds if j is an elementary embedding $j : V_\lambda \to V_\lambda$.

The axiom $\mathbf{I}_0$ is therefore simply $(\exists\lambda)(\exists j)\,\mathbf{I}_0(\lambda, j)$, and similarly for $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_E$, and $\mathbf{I}_3$.

Theorem 6.2.2. *The axiom* $\mathbf{I}_0$ *implies* $\mathbf{I}_1$*,* $\mathbf{I}_1$ *implies* $\mathbf{I}_2$*,* $\mathbf{I}_2$ *implies* $\mathbf{I}_E$*, and* $\mathbf{I}_E$ *implies* $\mathbf{I}_3$*.*

Proof. Only two of the implictions are non-trivial: $\mathbf{I}_1$ implies $\mathbf{I}_2$ and $\mathbf{I}_2$ implies $\mathbf{I}_E$. For a proof that $\mathbf{I}_1$ implies $\mathbf{I}_2$, see Proposition 24.2 in [Kan94].

To see that $\mathbf{I}_2$ implies $\mathbf{I}_E$, let λ and j' witness $\mathbf{I}_2$, and set $j = j'\restriction_{V_\lambda}$. Set $j_0 = j$, and $j_{i+1} = j_i(j_i)$; then, one has the sequence of embeddings below:

$$V_\lambda \xrightarrow{j_0} V_\lambda \xrightarrow{j_1} V_\lambda \xrightarrow{j_2} \cdots$$

Recall that $j_{n,m}$ denotes the composite map between the n-th copy of V_λ and the m-th copy, so that $j_0 = j_{0,1}$. Further recall that, letting $\mathcal{M} = \varinjlim\langle V_\lambda, j_i\rangle$, we denote by $j_{n,\omega}$ the canonical elementary embedding from the n-th copy of V_λ to $\mathcal{M}$.

Then, suppose $\mathcal{M}$ were ill-founded, i.e., there were some ordinal α such that $V \models$ "$(j_{0,\omega}(\alpha), \in_{\mathcal{M}})$ is ill-founded". Let α_0 be the least ordinal such that its image is ill-founded. It follows that

$$V \models \text{“}(\forall\beta < \alpha_0),\ (j_{0,\omega}(\beta), \in_{M_\omega}) \text{ is well-founded”} \tag{6.1}$$

and so, by the elementariness of j', applying j' to Statement 6.1, we obtain

$$\mathcal{M} \models \text{“}(\forall\beta < j(\alpha_0)),\ (j'(j_{0,\omega})(\beta), \in_{j'(M_\omega)}) \text{ is well-founded”.} \tag{6.2}$$

Now, note that since $j = j'\restriction_{V_\lambda}$ and $j(j_n) = j_{n+1}$, $j'(j_{0,\omega})$ is, by definition, the map from the first copy of V_λ in the system

$$V_\lambda \xrightarrow{j_1} V_\lambda \xrightarrow{j_2} V_\lambda \xrightarrow{j_3} \cdots$$

which is precisely $j_{1,\omega}$; moreover, we see that its direct limit is also $\mathcal{M}$ by an easy application of the universal property of direct limits. Now, being well-founded is an absolute property for inner models; hence,

$$V \models \text{“}(\forall\beta < j(\alpha_0)),\ \big((j_{1,\omega})(\beta), \in_{M_\omega}\big) \text{ is well-founded”.} \tag{6.3}$$

Continuing in this way, we see that

$$V \models \text{“}(\forall\beta < j_n(\alpha_0)),\ \big((j_{n,\omega})(\beta), \in_{M_\omega}\big) \text{ is well-founded”} \tag{6.4}$$

for all $n < \omega$. Therefore, it follows that for no $\beta \in_{\mathcal{M}} j_{0,\omega}(\alpha)$, $\beta \in \mathrm{ORD}^{\mathcal{M}}$ does $V \models$ "$(\beta, \in_{\mathcal{M}})$ is ill-founded". Hence $j_{0,\omega}(\alpha)$ is the $\in_{\mathcal{M}}$-least ill-founded ordinal, contrary to the fact that there can be no least ill-founded ordinal. Therefore $\mathcal{M}$ is, in fact, well-founded. □

Remark 6.2.3. Somewhat more is actually true. Each of the axioms $\mathbf{I}_1$, $\mathbf{I}_2$, $\mathbf{I}_E$, and $\mathbf{I}_3$ is known to *strongly* imply the next weakest axiom; that is, if λ is the least cardinal satisfying $\mathbf{I}_1$, then the least cardinal λ' satisfying $\mathbf{I}_2$ is less than $\mathbf{I}_1$. See [Lav97].

6.3 Elementary Embeddings

We require the following lemma.

Lemma 6.3.1. *If $j : V_\lambda \to V_\lambda$ has an extension to an elementary embedding $j^+ : V_{\lambda+1} \to V_{\lambda+1}$, then that extension is unique.*

Proof. Note that for any $X \subset V_\lambda$, $X = \bigcup_{\beta<\lambda}(X \cap V_\beta)$. Therefore,

$$j^+(X) = j^+\left(\bigcup_{\beta<\lambda}(X \cap V_\beta)\right) = \bigcup_{\beta<\lambda}(j^+(X) \cap V_\beta) = \bigcup_{\beta<\lambda}(j(X) \cap V_\beta) \tag{6.5}$$

and hence j^+, if it exists, is uniquely determined by j. □

The following strengthening of Lemma 6.3.1 also holds. Its proof, while itself simple, relies upon a technical lemma.

Proposition 6.3.2. *Suppose $j : V_\lambda \to V_\lambda$ has an extension to an embedding $j : L_\alpha(V_{\lambda+1}) \to L_\beta(V_{\lambda+1})$, for ordinals $\alpha, \beta \leq \lambda$ such that α and β are limit ordinals. Then, that extension is unique.*

Lemma 6.3.3. *Let, for $\alpha \leq \lambda$, α a limit ordinal,*

$$X_\alpha = \Big\{a \in L_\alpha(V_{\lambda+1}) : \big(\exists\varphi(\boldsymbol{x}, \vec{\boldsymbol{y}}) \in \text{FORM}\big)\big(\exists\vec{p} \in {}^n(V_{\lambda+1})\big) \\ \big(L_\alpha(V_{\lambda+1}) \models \varphi(a, \vec{p}) \wedge (\forall\boldsymbol{x})(\varphi(\boldsymbol{x}, \vec{p}) \to \boldsymbol{x} = a)\big)\Big\} \tag{6.6}$$

i.e., those sets definable in $L_\alpha(V_{\lambda+1})$ from parameters in $V_{\lambda+1}$. Then $X_\alpha = L_\alpha(V_{\lambda+1})$.

Sketch of Proof of Lemma 6.3.3. It is not difficult to show that X_α is an elementary substructure of $L_\alpha(V_{\lambda+1})$; see Lemma 1 in [Lav01] for a proof. Since $V_{\lambda+1} \in X_\alpha$ and $V_{\lambda+1} \subset X_\alpha$, the function mapping β to $L_\beta(V_{\lambda+1})$ for $\beta < \alpha$ is definable in X_α, whence equality follows by absoluteness. □

Remark 6.3.4. We actually can prove the following strengthening of Lemma 6.3.3: every element of $L_\alpha(V_{\lambda+1})$ is definable using a *single* parameter in $V_{\lambda+1}$. Since $V_{\lambda+1}$ does not satisfy the Axiom of Pairing, this is not immediately obvious; however, one could, for instance, code the pair $\{x, y\}$, $x, y \in V_{\lambda+1}$ as $\{x \cap V_{\kappa_i}, y \cap V_{\kappa_i} : i < \omega\}$ and then reconstruct the separate parameters.

In general, certain standard constructions, such as the formation of pairs or of direct limits, cannot be carried explicitly in $V_{\lambda+1}$. Nevertheless, it is possible to code these sets as subsets of $V_{\lambda+1}$, as above, and in the sequel we shall do so without comment.

Remark 6.3.5. It is actually possible to show that there is a *single* formula $\theta(\mathbf{x}, \mathbf{y})$ such that for every $a \in L_\alpha(V_{\lambda+1})$ there is some $p \in V_{\lambda+1}$ such that $L_\alpha(V_{\lambda+1}) \models \theta(a, p)$ but $L_\alpha(V_{\lambda+1}) \not\models \theta(b, p)$ for every $b \neq a$ in $L_\alpha(V_{\lambda+1})$.

Proof of Propositon 6.3.2. With Lemma 6.3.3, the proof becomes essentially trivial. Let $a \in L_\alpha(V_{\lambda+1})$ be defined by the formula $\varphi(\mathbf{x}, \mathbf{y})$ and the parameter $p \in V_{\lambda+1}$ so that

$$L_\alpha(V_{\lambda+1}) \models \varphi(a) \wedge (\forall\mathbf{x})(\varphi(\mathbf{x}, p) \to \mathbf{x} = a). \tag{6.7}$$

Let $j : L_\alpha(V_{\lambda+1}) \to L_\beta(V_{\lambda+1})$ be such that $j|_{V_\lambda+1} = j'|_{V_\lambda+1}$. Then,

$$L_\beta(V_{\lambda+1}) \models \varphi(j(a)) \wedge (\forall \mathbf{x})(\varphi(\mathbf{x}, j(p)) \to \mathbf{x} = j(a)) \tag{6.8}$$

and also

$$L_\beta(V_{\lambda+1}) \models \varphi(j'(a)) \wedge (\forall \mathbf{x})(\varphi(\mathbf{x}, j'(p)) \to \mathbf{x} = j(a)). \tag{6.9}$$

However, since $j(p) = j'(p)$, it follows that $j'(a) = j(a)$. □

Remark 6.3.6. As a consequence of Proposition 6.3.2 and Lemma 6.3.1, one need not distinguish between an elementary embedding's restriction and extension, since they both uniquely determine each other. Therefore, for $j : L_\alpha(V_{\lambda+1}) \to L_\beta(V_{\lambda+1})$ $\alpha \le \beta \le \lambda$, we shall not distinguish between j, $j\restriction_{L_{\alpha'}(V_{\lambda+1})}$ for $\alpha' < \alpha$, $j\restriction_{V_{\lambda+1}}$, and $j\restriction_{V_\lambda}$ unless necessary to disambiguate.

Definition 6.3.7 (α-Embeddings)**.** For $\alpha < \lambda$, α a limit ordinal, we say that $k : V_\lambda \to V_\lambda$ is an *α-embedding* if k extends to an embedding $k^+ : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$. Note that necessarily the critical point of k is greater than α. By Proposition 6.3.2, this extensions are unique.

Similarly, say that $k : V_\lambda \to V_\lambda$ is a *λ-embedding* if k extends to an embedding $k^\dagger : L_\lambda(V_{\lambda+1}) \to L_\lambda(V_{\lambda+1})$.

In the case of embeddings $j, k : V_\lambda \to V_\lambda$, it is well-known that

$$j(k) = \bigcup_{\alpha<\lambda} j(k \cap V_\alpha) \tag{6.10}$$

is likewise an elementary embedding from V_λ to V_λ. The proof is similar to the following analogous result for $j, k \in \mathcal{E}_\alpha$, $\alpha \le \lambda$.

Lemma 6.3.8. *Let $j, k \in \mathcal{E}_\alpha$, $\alpha \le \lambda$. Then, define*

$$j(k) = \bigcup_{\beta<\alpha} j(k \cap L_\beta(V_{\lambda+1})). \tag{6.11}$$

Then, $j(k) \in \mathcal{E}_\alpha$ as well.

Proof of Lemma 6.3.8. We begin with the observation that for $\beta < \alpha$, the extension of k to $L_\beta(V_{\lambda+1})$—which we shall denote by k_β—is an element of $L_\alpha(V_{\lambda+1})$. This follows since, by Remark 6.3.5, there is a formula expressing the fact that $k(a) = b$ for $a, b \in L_\alpha(V_{\lambda+1})$.

Note that an embedding $h : L_\alpha(V_{\lambda+1}) \to L_\alpha(V_{\lambda+1})$ is elementary if and only if, for every $\varphi(\mathbf{x}) \in \textsc{Form}$, h preserves *φ-truth*. For a given $\varphi \in \textsc{Form}$, h is said to preserve φ-truth if for any parameter $p \in L_\alpha(V_{\lambda+1})$,

$$L_\alpha(V_{\lambda+1}) \models \varphi(\mathbf{x})[p] \iff L_\alpha(V_{\lambda+1}) \models \varphi(\mathbf{x})[h(p)]. \tag{6.12}$$

Note that while elementariness is a second-order property, for any fixed φ, φ-truth is a first-order property, and trivially formalizable in $\mathcal{L}_\in$.

Suppose that $\alpha < \lambda$, so that $\alpha < \textsc{Crt}(j)$. Now, note that for $\beta < \alpha$,

$$L_\alpha(V_{\lambda+1}) \models (\forall \mathbf{x} \in L_\beta(V_{\lambda+1}))(\varphi(\mathbf{x}) \iff \varphi(k_\beta(\mathbf{x}))). \tag{6.13}$$

Applying j yields

$$L_\alpha(V_{\lambda+1}) \models (\forall \mathbf{x} \in L_\beta(V_{\lambda+1}))(\varphi(\mathbf{x}) \iff \varphi(j(k_\beta)(\mathbf{x}))). \tag{6.14}$$

and hence it follows that $j(k) = \bigcup_{\beta<\alpha} j(k_\beta)$ preserves φ-truth. Since $\varphi(\mathbf{x})$ was arbitrary, it follow that $j(k)$ is an α-embedding.

The proof for $\alpha = \lambda$ is, in all essential details, identical. □

Remark 6.3.9. By Lemma 6.3.8, it follows that $\mathcal{E}_\alpha$ has an algebraic structure for every $\alpha \leq \lambda$. Namely, $\mathcal{E}_\alpha$ is closed under composition—j and k map to $j \circ k$—and application—j and k map to $j \cdot k = j(k)$. Moreover, one can easily verify the following algebraic relations for $i, j, k \in \mathcal{E}_\alpha$:

(i) $i \circ (j \circ k) = (i \circ j) \circ k$, $(i \circ j) \cdot k = i \cdot (j \cdot k)$, $i \cdot (j \circ k) = (i \cdot j) \circ (i \cdot k)$, and $i \circ j = (i \cdot j) \circ i$;

(ii) $i \cdot (j \cdot k) = (i \cdot j) \cdot (i \cdot k)$.

Under application, $\mathcal{E}_\alpha$ forms a free left-distributive algebra. (See [Kan12, p. 329].)

Lemma 6.3.10. *If $j, k \in \mathcal{E}_\alpha$, then* $\textsc{Crt}(j(k)) = j(\textsc{Crt}(k))$.

Proof. Simply note that, irrespective of our choice of $\beta < \alpha$,

$$L_\alpha(V_{\lambda+1}) \models (\forall \gamma < \textsc{Crt}(k_\beta))(k_\beta(\gamma) = \gamma) \tag{6.15}$$

and apply j. □

6.4 Measures and Restricted Ultraproducts

Definition 6.4.1 (Towers of Measures)**.** Suppose one has a sequence of the form $\langle X_i : i < \omega \rangle$, where each X_i is a set with an associated 0-1 measure μ_i defined on the σ-algebra $\mathscr{B}_i$ along with maps $\pi_i : X_{i+1} \to X_i$, i.e.,

$$X_0 \xleftarrow{\pi_0} X_1 \xleftarrow{\pi_1} X_2 \xleftarrow{\pi_2} X_3 \xleftarrow{\pi_3} \cdots \tag{6.16}$$

such that for all i,

(i) The pushforward measure of μ_{i+1} by π_i equals μ_i, i.e., for all $S \in \mathscr{B}_i$, $\mu_i(S) = \mu_{i+1}(\pi_i^{\textsc{Pre}}(S))$;

(ii) The pullback of μ_i by π is equals μ_{i+1}, i.e., for all $S \in \mathscr{B}_i$, $\mu_{i+1}(S) = \mu_i(\pi_i``S)$.

Then we say that $\langle X_i : i < \omega \rangle$ defines a *tower of measures.*

We say that the tower $\langle X_i : i < \omega \rangle$ is *countably complete* if whenever for a sequence of measure one sets $\langle S_n : n < \omega \rangle$, $S_i \in \mathscr{B}_i$, there exist a sequence $\langle x_n : n < \omega \rangle$, $x_i \in S_i$, such that $\pi_i(x_{i+1}) = x_i$ for all $i < \omega$. Such a collection $\langle x_i : i < \omega \rangle$ is called a *thread.*

Remark 6.4.2. Let $(X, \mu, \mathscr{B})$ be a 0-1 measure space. Given a model $\mathcal{M}$, one can always define the reduced product

$$\prod_{x \in X} \mathcal{M}/\mathscr{B}$$

where $f \sim g$ for $f, g \in \prod_{x \in X} \mathcal{M}$ if and only if $\mu\{x : f(x) = g(x)\} = 1$, and $\prod_{x \in X} \mathcal{M}/\mathscr{B} \models [f] \in_\mu [g]$ if and only if $\mu\{x \in X : f(x) \in g(x)\} = 1$, with the modeling relation for formulas of greater complexity defined in the usual inductive manner.

Since a measure is "close" to an ultrafilter, one would hope that something close to Łos's Theorem would hold. Unfortunately, reduced products in general contain too many sets for this to be the case, since, for instance $\prod_{x \in X} M_x/\mathscr{B}$ may model $\neg([f] \in_\mu [g])$ simply because neither $\{x \in X : f(x) \in g(x)\}$ nor $\{x \in X : f(x) \notin g(x)\}$ is measurable. Following Martin's proof in Subsection 5.4.9 of Chapter 5, we would like to study a certain kind of game played with reference to a canonical form for the projective sets. However, due to extra technical requirements, we cannot define actual ultrafilters. One can remedy this problem by restricting the functions we allow to represent sets in the reduced product, which motivates the following definition.

Definition 6.4.3 (Restricted Ultraproduct)**.** Let $(X, \mu, \mathscr{B})$ be a 0-1 measure space and let $\mathcal{M}$ be given. Suppose

$$\mathcal{F} \subset \prod_{x \in X} \mathcal{M} \tag{6.17}$$

is a family of functions such that two conditions hold:

(i) For all $f, g \in \mathcal{F}$, the sets $\{x \in X : f(x) = g(x)\}$ and $\{x \in X : f(x) \in g(x)\}$ are in $\mathscr{B}$;

(ii) For any set of non-empty definable subsets of $\mathcal{M}$, $\{Y_x\}_{x \in X}$, there is some $f \in \mathcal{F}$ such that $f(x) \in Y_x$ for almost every $x \in X$.

Under these circumstances, we say that $\mathcal{F}$ is *adequate* family of sets.

Given a 0-1 measure space $(X, \mu, \mathscr{B})$ and an adequate family of sets $\mathcal{F}$, we say that the set

$$M = \left[\mathcal{F} \cap \prod_{x \in X} \mathcal{M}\right] \Big/ \sim \tag{6.18}$$

is a *restricted ultraproduct*, where $f \sim g$ precisely when $\mu\{x \in X : f(x) = g(x)\} = 1$, and denoting by $[f]$ the equivalence class of f, $[f] \in_M [g]$ if and only if $\mu\{x \in X : f(x) \in g(x)\} = 1$.

For brevity, we shall also write

$$\prod_{x \in X} \mathcal{M}/\mathscr{B} \tag{6.19}$$

when the family of functions $\mathcal{F}$ is clear from context.

Lemma 6.4.4. *Restricted ultraproducts satisfy Łos's Theorem; that is, for any formula φ, if $M = \prod_{x \in X} \mathcal{M}/\mathscr{B}$ is a restricted ultrapower over the measure space*

$(X, \mu, \mathscr{B})$ and $\mathcal{F}$ is an adequate family of functions, then for all $(f_1, \ldots, f_n) \in {}^n\mathcal{F}$ and any formula φ with less than n free variables,[2]

$$M \models \varphi\,[f_0, \ldots, f_{n-1}] \iff \mu\{x \in X : \mathcal{M} \models \varphi[f_1(x), \ldots, f_n(x)]\} = 1. \tag{6.20}$$

Proof. Since $\mathcal{F}$ is adequate, Equation 6.20 holds for all φ where φ is an atomic formula. Therefore, suppose for induction that 6.20 holds for φ and ψ with any choice of parameters $(f_0, \ldots, f_n)$. Let $[\![\varphi, f_0, \ldots, f_n]\!]_X = \{x \in X : \mathcal{M} \models \varphi[f_0(x), \ldots, f_n(x)]\}$.

Then, it immediately follows that

$$[\![\neg\varphi, f_0, \ldots, f_n]\!]_X = X - [\![\varphi, f_0, \ldots, f_n]\!]_X \tag{6.21}$$

and

$$[\![\varphi \wedge \psi, f_0, \ldots, f_n]\!]_X = [\![\varphi, f_0, \ldots, f_n]\!]_X \cap [\![\psi, f_0, \ldots, f_n]\!]_X \tag{6.22}$$

Hence, in this notation, Equation 6.20 merely states the property

$$M \models \chi\,[g_0, \ldots, g_m] \iff \mu[\![\chi, g_0, \ldots, g_m]\!]_X = 1. \tag{6.23}$$

It follows that

$$\begin{aligned}
M \models \neg\varphi\,[f_0, \ldots, f_n] &\iff M \not\models \varphi\,[f_0, \ldots, f_n] && (6.24)\\
&\iff \mu[\![\varphi, f_0, \ldots, f_n]\!]_X \neq 1 && (6.25)\\
&\iff \mu[\![\varphi, f_0, \ldots, f_n]\!]_X = 0 && (6.26)\\
&\iff \mu[\![\neg\varphi, f_0, \ldots, f_n]\!]_X = 1 && (6.27)
\end{aligned}$$

and similarly for $M \models \varphi \wedge \psi[f_0, \ldots, f_n]$.

Next, note that for each $x \in X$, the set $\{y \in \mathcal{M} : \mathcal{M} \models \varphi[y, f_1(x), \ldots, f_n(x)]\}$ is a definable subset. If it is non-empty, set $Y_x = \{y \in \mathcal{M} : \mathcal{M} \models \varphi[y, f_1(x), \ldots, f_n(x)]\}$. If it is empty, set $Y_x = \mathcal{M}$. Then, it is clear that

$$\begin{aligned}
M \models (\exists x)\varphi(x)\,[f_1, \ldots, f_n] &\iff (\exists f_0 \in \mathcal{F})\,(M \models \varphi\,[f_0, f_1, \ldots, f_n]) && (6.28)\\
&\iff \mu[\![\varphi, f_0, \ldots, f_n]\!]_X = 1 && (6.29)\\
&\iff \mu[\![(\exists x)\varphi(x), f_1, \ldots, f_n]\!]_X = 1 && (6.30)
\end{aligned}$$

where the last equivalence follows by Condition (ii) of Definition 6.4.3 and the fact that

$$\mu\big([\![(\exists x)\varphi(x), f_1, \ldots, f_n]\!]_X \,\triangle\, [\![\varphi, f_0, \ldots, f_n]\!]_X\big) = 0 \tag{6.31}$$

□

Corollary 6.4.5. *For the model $M = \prod_{x\in X} \mathcal{M}/\mathscr{B}$, the relation $\in_M$ is well-founded. Since the contant functions $c_m : x \mapsto m$ for all $m \in \mathcal{M}$ are necessarily in $\mathcal{F}$ by Condition (ii) of Definition 6.4.3, one can form the canonical injection $\iota : \mathcal{M} \to M$ by $\iota : m \mapsto [c_m]$; then ι is an elementary embedding.*

Let $\mathrm{ULT}_0(\mathcal{M}, \mu)$ denote the Mostowski collapse of $\mathcal{M}$. Then there is an elementary embedding $\iota_\mu : \mathcal{M} \to \mathrm{ULT}_0(\mathcal{M}, \mu)$ equal to the composition of ι and the Mostowski collapse. Under these circumstances, we call $\mathrm{ULT}_0(\mathcal{M}, \mu)$ the restricted ultrapower *of $\mathcal{M}$.*

[2]Properly, one ought to write $\varphi[[f_0], [f_1], \ldots, [f_{n-1}]]$ rather than $\varphi[f_0, f_1, \ldots, f_{n-1}]$; however, since the satisfaction relation is not affected by which representative of the equivalence class one chooses, we opt for the latter notation.

Proof. Note that the filter of μ-measure one sets is σ-complete by definition. Therefore, the proof that M is well-founded is exactly as in the case of a standard ultrapower; see Theorem 5.3 in [Kan94].

Since $\{x \in X : \mathcal{M} \models \sigma\}$ is, for every sentence σ, either X or $\emptyset$, it follows by Lemma 6.4.4 that ι_μ is an elementary embedding. $\square$

Definition 6.4.6 (Strong Well-Foundedness)**.** A model $(\mathcal{M}, \in_{\mathcal{M}})$ is *strongly well-founded* if and only if from

$$\mathcal{M} \models \text{“}R \text{ is a well-founded relation”} \tag{6.32}$$

it follows that

$$V \models \text{“}(R, \epsilon_{\mathcal{M}}) \text{ is a well-founded relation”}. \tag{6.33}$$

Remark 6.4.7. The purpose of the foregoing discussion has been to allow us to establish the following lemma, which will play a critical role in the proof of Theorem 6.7.1.

Lemma 6.4.8. *Let $\langle X_i : i < \omega\rangle$ be a tower of measures where $X_i \in L_{\omega_1}(V_{\lambda+1})$ for each $i < \omega$, and μ_i, the measure associated to X_i, has as its domain the σ-algebra $\mathscr{B}_i = \mathcal{P}(X_i) \cap L_{\omega_1}(V_{\lambda+1})$. Let $\mathcal{F}_i = L_{\omega_1}(V_{\lambda+1}) \cap {}^{X_i}(V_{\lambda+1})$ be an adequate family of functions for $(X_i, \mu_i, \mathscr{B}_i)$.*

In particular, the maps $\pi_i : X_{i+1} \to X_i$ give rise to canonical maps

$$j_i : \text{ULT}_0(V_{\lambda+1}, \mu_i) \to \text{ULT}_0(V_{\lambda+1}, \mu_{i+1}) \tag{6.34}$$

as in the diagram below:

$$\text{ULT}_0(V_{\lambda+1}, \mu_0) \xrightarrow{j_0} \text{ULT}_0(V_{\lambda+1}, \mu_1) \xrightarrow{j_1} \text{ULT}_0(V_{\lambda+1}, \mu_3) \xrightarrow{j_2} \cdots \tag{6.35}$$

Then, the direct limit of Diagram 6.35 above is well-founded if and only if the tower of measures giving rise to it is countably complete.

Proof. Note that if $g : X_i \to V_{\lambda+1}$, then $g \circ \pi_i : X_{i+1} \to V_{\lambda+1}$. Define, therefore $j_i : g \mapsto g \circ \pi_i$. We seek to show that j_i is an elementary embedding. However, note that for any formula φ and set of parameters $g_0, \ldots, g_n : X_i \to V_{\lambda+1}$,

$$\begin{aligned}\pi_i^{\text{PRE}}(&[\![\varphi, g_0, \ldots, g_n]\!]_{X_i}) \\ &= [\![\varphi, g_0 \circ \pi_i, \ldots, g_n \circ \pi_i]\!]_{X_{i+1}} = [\![\varphi, j_i(g_0), \ldots, j_i(g_n)]\!]_{X_{i+1}}\end{aligned} \tag{6.36}$$

and hence, by Lemma 6.4.4, it immediately follows that j_i is an elementary embedding, since $\mu_{i+1}(\pi_i^{\text{PRE}}(S)) = 1$ if and only if $\mu_i(S) = 1$.

Let $M = \varinjlim(\text{ULT}_0(X_i, \mu_i), j_i)$. First, suppose that the tower of measures $\langle X_i : i < \omega\rangle$ were countably complete, but M were not strongly well-founded. Then, let $(R', \in_M)$ be some ill-founded relation such that $M \models$ “R' is well-founded”. Since λ^+ is regular, it is not difficult to construct a well-founded relation R in $V_{\lambda+1}$ such that $R' \subset j_{0,\omega}(R)$. Without loss of generality, there would have to exist a sequence $\langle f_i : i < \omega\rangle$ such that $[\![(\mathbf{x}, \mathbf{y}) \in j_{0,i+1}(R), j_i(f_i), f_{i+1}]\!]_{X_{i+1}}$ has measure one for all $i < \omega$. Let $\langle x_i : i < \omega\rangle$ be a thread. Then $(f(x_i), f_{i+1}(x_{i+1})) \in R$ for all $i < \omega$, since $j_{m,n}(f_m)(x_n) = f_m(x_m)$ for all $m < n < \omega$. Therefore, R is not well-founded, contrary to assumption.

Lastly, we show that if $\langle X_i : i < \omega \rangle$ is not countably complete, then M is not well-founded, and hence, *a fortiori*, not strongly well-founded. Let $\langle S_i : i < \omega \rangle$ be a sequence of measure one sets with no thread. Set

$$f_k(x) = \begin{cases} n - k & x \in \left(\bigcap_{i<n} \pi_{i,k} \text{``} S_i \right) \setminus \pi_{n,k} \text{``} S_n \\ 0 & \text{else} \end{cases} \tag{6.37}$$

Trivially, $f_n \in L_{\omega_1}(V_{\lambda+1})$ for all n. Moreover, $[\![\mathbf{x} \in \mathbf{y}, j_i(f_i), f_{i+1}]\!]_{X_{i+1}}$ has measure one, since for every $x \in \bigcap_{i \le n < \omega} \pi_{n,i} \text{``} S_n$, there exists some $m$ such that $x \neq \pi_{m,i}(x')$ for every $x' \in S_m$. Now, $\mu_i \left( \bigcap_{i \le n < \omega} \pi_{n,i} \text{``} S_n \right) = 1$, so it follows that

$$M \models j_{i+1,\omega}(f_{i+1}) \in j_{i,\omega}(f_i) \tag{6.38}$$

for all i; hence M is not well-founded. □

6.5 Representations of Point Sets

Definition 6.5.1 (A-Representations)**.** Let $\delta < \lambda$ be fixed in advance. Call any function $\pi : {}^{<\omega}\omega \to \mathcal{E}_\lambda$ a *representation.* We say that a representation is *good* when the following two conditions hold:

(i) For any $x = (x_0, x_1, \ldots) \in \mathcal{N}$, if we let $j_0 = \pi(\emptyset)$, $j_1 = \pi(x_0)$, $j_2 = \pi(x_0, x_1)$, and so on, then for any $n < \omega$, $\textsc{Crt}(j_n) < \sup j_{0,n} \text{``} \delta$;

(ii) For all $x = (x_0, x_1, \ldots) \in \mathcal{N}$, defining $\pi(x)$ to be the direct limit of the system

$$V_{\lambda+1} \xrightarrow{\pi(x \restriction 0)} V_{\lambda+1} \xrightarrow{\pi(x \restriction_1)} V_{\lambda+1} \xrightarrow{\pi(x \restriction_2)} \cdots \tag{6.39}$$

then *either* $\pi(x)$ is *ill-founded below* δ, i.e., there exists $\alpha < \delta$ such that $j_{0,\omega}(\alpha)$ is an ill-founded ordinal in $\pi(x)$; *or* $\pi(x)$ is strongly well-founded.

When π is a good representation, then we say that the set

$$A_\pi = \{ x \in \mathcal{N} : \pi(x) \text{ is ill-founded below } \delta \} \tag{6.40}$$

has an *A-representation* given by π.

Definition 6.5.2 (B-Representations)**.** Let $\delta < \lambda$ again be fixed in advance, and let $\pi : {}^{<\omega}\omega \to \mathcal{E}_\lambda$ be an arbitrary representation, not necessarily good. Then we say that the set

$$B_\pi = \{ x \in \mathcal{N} : \pi(x) \text{ is not strongly well-founded} \}. \tag{6.41}$$

has a *B-representation* given by π.

Remark 6.5.3. Note that π is a good representation precisely when $A_\pi = B_\pi$.

Definition 6.5.4 (β-Sequences)**.** Let j_0, j_1, … be a sequence of embeddings in $\mathcal{E}_\lambda$ such that

$$V_{\lambda+1} \xrightarrow{j_0} V_{\lambda+1} \xrightarrow{j_1} V_{\lambda+1} \xrightarrow{j_2} \cdots \tag{6.42}$$

is ill-founded below δ. Suppose $\langle \beta_i : i < \omega \rangle$ is a sequence of limit ordinals such that $\delta > \beta_0$ and $j_i(\beta_i) > \beta_{i+1}$ for every $i < \omega$. Then, we say that $\langle \beta_i : i < \omega \rangle$ is a *β-sequence.*

Remark 6.5.5. A β-sequence $\langle \beta_i : i \in \omega \rangle$ is useful because it witnesses as quickly as possible that a given direct limit is ill-founded below δ.

Lemma 6.5.6. *Let j_0, j_1, ... be a sequence of embeddings in $\mathcal{E}_\lambda$ such that*

$$V_{\lambda+1} \xrightarrow{j_0} V_{\lambda+1} \xrightarrow{j_1} V_{\lambda+1} \xrightarrow{j_2} \cdots \tag{6.43}$$

is ill-founded below δ. Then there is a β-sequence $\langle \beta_i : i < \omega \rangle$ witnessing ill-foundation below δ.

Proof. Let $\eta_0 < \delta$ be a limit ordinal such that $j_{0,\omega}(\eta_0)$ is ill-founded. Then, there exist n_0 and η_1 such that $j_{n_0,\omega}(\eta_1)$ is ill-founded, and $j_{0,n_0}(\eta_0) > \eta_1 + \omega^2$. Then, set

$$\beta_0 = \eta_0 + n_0\omega \quad \beta_1 = j_0(\eta_0) + (n_0 - 1)\omega \quad \ldots \quad \beta_{n_0-1} = j_{0,n_0}(\eta_0) + \omega. \tag{6.44}$$

Then, there further exist n_1 and η_2 such that $j_{n_2,\omega}(\eta_2)$ is is ill-founded, and moreover $j_{n_1,n_2}(\eta_1) > \eta_2 + \omega^2$. Then, set

$$\beta_{n_0} = \eta_1 + n_1\omega \quad \beta_{n_0+1} = j_{n_0}(\eta_1) + (n_1 - 1)\omega \quad \ldots \quad \beta_{n_0-1} = j_{n_0,n_1}(\eta_1) + \omega. \tag{6.45}$$

Continuing in this manner, we build the desired β-sequence. □

6.6 Strongly Well-Founded Direct Limits

For the remainder of this section, fix $\delta < \lambda$ and $J' \in \mathcal{E}_\lambda$ such that $\textsc{Crt}(J') = \delta$, as well as a sequence of embeddings $\langle j_i : i < \omega \rangle \in {}^{\omega}(\mathcal{E}_\lambda)$. Let $\mathcal{M}$ denote the direct limit of

$$V_{\lambda+1} \xrightarrow{j_0} V_{\lambda+1} \xrightarrow{j_1} V_{\lambda+1} \xrightarrow{j_2} \cdots \tag{6.46}$$

and let $j_{m,\omega}$ be the canonical elementary embedding from the m-th copy of $V_{\lambda+1}$ to $\mathcal{M}$.

Let $X_n = {}^{n}(\mathcal{E}_0)$, let π_n be the canonical projection from X_{n+1} onto X_n, and let $\mathscr{B}_n = L_{\omega_1}(V_{\lambda+1}) \cap \mathcal{P}(X_n)$. Then, we define a tower of measures ν, where

$$(\nu_1, X_1, \mathscr{B}_1) \xleftarrow{\pi_1} (\nu_2, X_2, \mathscr{B}_2) \xleftarrow{\pi_2} (\nu_3, X_3, \mathscr{B}_3) \xleftarrow{\pi_3} \cdots \tag{6.47}$$

To do so, we define ν_m inductively. Let $\nu_1(S) = 1$ if and only if $J' \in J'(S)$; note that J is a λ-embedding, so S is an the range of (an extension of) J, and so ν_0 is well-defined. Set $j_0^* = k_0^0 = J'$. Then, suppose that ν_m, j_{m-1}^*, and $\vec{k}_m = (k_0^{m-1}, k_1^{m-1}, \ldots, k_{m-1}^{m-1})$ have all been defined, and furthermore that $k_{m-1}^{m-1} = j_{m-1}^*$. Set

$$j_m^* = \left[j_{0,m}^*(j_m)\right](j_{m-1}^*). \tag{6.48}$$

Note that by inductively applying Lemma 6.3.8, it folows that j_m^* is a λ-embedding. Then, it is possible to set, for $S \in \mathscr{B}_{m+1}$,

$$\nu_{m+1}(S) = 1 \iff (j_m^*(k_0^{m-1}), \ldots, j_m^*(k_{m-1}^{m-1}), j_m^*) \in j_{0,m+1}^*(S). \tag{6.49}$$

We refer to $\vec{k}_m = (k_0^{m-1}, k_1^{m-1}, \ldots, k_{m-1}^{m-1})$ as the *generic point*[3] of ν_m.

[3]In general, there is no notational clash between this notion of a generic point, and standard usage, in which a generic point is said to have a property P if P holds of all elements of a measure one set. For, if $\varphi(\mathbf{x})[p]$ expresses the property P, then P holds of a ν_m measure-one set if and only if P' holds of $\vec{k}_m$, where P' is the property expressed by $\varphi(\mathbf{x})[j_{0,\omega}(p)]$.

It follows immediately from Equation 6.49 and the σ-completeness of $\mathscr{B}_m$ that ν_m does indeed define a measure.

The tower of measures ν has two important properties, given in the following two lemmata.

Lemma 6.6.1. *A ν_m-generic m-tuple $\vec{k} = (k_0, \ldots, k_{m-1})$ in X_k has critical points forming an initial segment of a β-sequence for $\mathcal{M}$; that is,*

$$\delta > \textsc{Crt}(k_0) \quad j_0(\textsc{Crt}(k_0)) > \textsc{Crt}(k_1) \quad \ldots \\ \ldots \quad j_{m-2}(\textsc{Crt}(k_{m-2})) > \textsc{Crt}(k_{m-1}). \quad (6.50)$$

Proof. First, note that Relation 6.50 is trivial in the case that $m = 1$:

$$\nu_1\{k \in \mathcal{E}_0 : \delta > \textsc{Crt}(k)\} \iff J \in \{k \in \mathcal{E}_0 : J(\delta) > \textsc{Crt}(k)\} \quad (6.51)$$

and since $\delta = \textsc{Crt}(J')$, the right hand side clearly holds.

Suppose, for induction, that Relation 6.50 holds for m. To show that it holds for $m+1$, it suffices by Equation 6.49 to verify, for $i = 0, \ldots, m$ that

$$\left[j^*_{0,m+1}(j_i)\right](\textsc{Crt}(k^m_i)) > \textsc{Crt}(k^m_{i+1}). \quad (6.52)$$

Now, for $i = 0, \ldots, m-1$, this follows since by the induction hypothesis,

$$\left[j^*_{0,m}(j_i)\right](\textsc{Crt}(k^{m-1}_i)) > \textsc{Crt}(k^{m-1}_{i+1}), \quad (6.53)$$

and hence applying j^*_m to both sides of Inequality 6.53 gives the desired result. For $i = m$, recall that

$$j^*_m = \left[j^*_{0,m}(j_{m-1})\right](j^*_{m-1}) \quad (6.54)$$

and further note that $k^{m-1}_{m-1} = j^*_{m-1}$. Therefore,

$$\left[j^*_{0,m+1}(j_m)\right]\left(j^*_m(k^{m-1}_{m-1})\right) = j^*_m\Big(\left[j^*_{0,m}(j_{m-1})\right](j^*_{m-1})\Big) = j^*_m(j^*_m), \quad (6.55)$$

and since $\textsc{Crt}(j^*_m) < j^*_m(\textsc{Crt}(j^*_m))$ by definition, the result follows. □

Lemma 6.6.2. *The tower ν is countably complete if and only if $\mathcal{M}$ is ill-founded below δ.*

Proof. First, suppose ν is countably complete. Let S_m be the set of m-tuples of embeddings whose critical points form an initial segment of a β-sequence of length m. Then, by Lemma 6.6.1, it follows that $\nu_m(S_m) = 1$. Therefore, if $\langle \vec{h}_i : i < \omega \rangle$ is a thread, then $\langle \textsc{Crt}(h_i) : i < \omega \rangle$ is the desired β-sequence, where

$$(h_0, h_1, \ldots) = \bigcup_{i<\omega} \vec{h}_i. \quad (6.56)$$

Next, suppose that $\mathcal{M}$ is ill-founded below δ, and let $\langle \beta_i : i < \omega \rangle$ be a witnessing β-sequence. Let $\langle S_i : i < \omega \rangle$ be a sequence of sets such that S_m has ν_m-measure one. We define the following two operations:

(i) *Operation I:* Given a set A of $m+1$-tuples of elements of $\mathcal{E}_0$, let A^* be the set of all $m+1$-tuples $\vec{h} = (h_0, \ldots, h_m)$ such that

(a) The embedding h_{m-1} is an element of $\mathcal{E}_\xi$ for $\xi < \omega_1$ greater than the least limit ordinal $\alpha < \omega_1$ such that $A \in L_\alpha(V_{\lambda+1})$;

(b) The fiber $A_{(h_0,\ldots,h_{m-1})}$ satisfies

$$j_{m-1}(h_{m-1}) \in \big[j_{m-1}(h_{m-1})\big](A_{(h_0,\ldots,h_{m-1})}); \tag{6.57}$$

(That is, $A_{(h_0,\ldots,h_{m-1})}$ has $\mu_{j_{m-1}(k_{m-1})}$-"measure" one, where $\mu_{j_{m-1}(h_{m-1})}$ is as in Equation 5.60, although in our case, it is not, properly speaking, a measure.)

(ii) *Operation II:* If A is a set of $m+1$-tuples, then A^* is π_m"A, a set of m-tuples.

Observe that if A^* is the result of applying Operation I to A, then $\nu_{m+1}(A^*) = \nu_m(A)$. Likewise, if A^* is the result of applying Operation II to A, then $\nu_m(A^*) = \nu_{m+1}(A)$.

Let Ξ be the closure of $\{S_i\}_{i<\omega}$ under Operations I and II. Then every $A \in \Xi$ has ν-measure one. Now, since $\textsc{Crt}(J) = \delta > \beta_0$, it follows that a ν_1-generic element of $\mathcal{E}_0$ has critical point greater than β_0. Therefore, since $|\Xi| = \aleph_0$, let h_0 be some element of A for all sets of singletons $A \in \Xi$. Consequently we can assume that h_0 is a β_0-embedding. Therefore, there is some k_1 such that $(h_0, h_1) \in A$ for all sets of duples $A \in \Xi$; this is possible since

$$j_0(h_0) \in \big[j_0(h_0)\big](A_{h_0}) \tag{6.58}$$

for all such A, and so A_{h_0} is non-empty. Moreover, since $j_0(h_0)$ is a $j_0(\beta_0)$-embedding, it follows that we can take h_1 to be a β_1-embedding, as $\beta_1 < j_0(\beta_0)$. Continuing in this manner, if we let $\mathfrak{h} = (h_0, h_1, \ldots)$, then $\langle \mathfrak{h}\restriction_i : i < \omega\rangle$ is exactly the desired thread. □

Remark 6.6.3. The connection between the proof of Theorem 6.7.1 and Theorem 5.4.9 is essentially contained in Lemma 6.6.2: we are here using the *methodology* of iterated product measures to establish countable completeness, even though we are unable to produce *actual* iterated product measures. Nevertheless, the upshot is the same.

6.7 Projective Determinacy

Theorem 6.7.1. *Suppose* $\mathbf{I}_0$ *holds, and let*

$$\Gamma = \{X \subset \mathcal{N} : X \textit{ has an } A\textit{-representation}\}. \tag{6.59}$$

Then Γ *is a* σ*-algebra closed under complements, countable unions, and projections. Moreover,* Γ *is determined. Therefore,* **PD** *holds.*

We shall prove Theorem 6.7.1 by a sequence of lemmas. First, we need the following lemma, which requires the full strength of $\mathbf{I}_0$.

Lemma 6.7.2. *There is some* $\delta < \lambda$ *such that* δ *is the critical point of a* λ*-embedding* J'*, and for all* $\alpha < \lambda^+$*, there is a* λ*-embedding* J *such that* $J_{0,\omega}(\eta) > \alpha$ *for some* $\eta < \delta$*.*

Proof. See Lemma 6.8.2 below. □

For the remainder of this section, the fixed δ of Definition 6.5.1 is the δ of Lemma 6.7.2

Lemma 6.7.3. *The point class* Γ *contains all open sets.*

Proof. Let $U \subset \mathcal{N}$ be an open set. We seek a representation $\pi : {}^{<\omega}\omega \to \mathcal{E}_\lambda$ such that $U = A_\pi$. Recall that for a sequence $s \in {}^{<\omega}\omega$, $O(s)$ denotes the open set of all $x \in \mathcal{N}$ such that $x\restriction_{|s|} = s$.

Let $J : V_{\lambda+1} \to V_{\lambda+1}$ be as in Lemma 6.7.2. Then, we can define

$$\pi(x\restriction_i) = \begin{cases} J_i & O(x \restriction i) - U \neq \emptyset \\ J & O(x \restriction i) - U = \emptyset \end{cases} \tag{6.60}$$

Then, it is not hard to see that A_π is an A-representation of U.

First, note that for $x \in -U$, for all i, $O(x\restriction_i) \cap -U \neq \emptyset$, since $-U$ is closed; hence, $\pi(x) = \varinjlim(V_{\lambda+1}, J_i)$, and the latter is strongly well-founded, since J is iterable by Theorem 6.2.2.

Next, suppose $x \in U$. Let $\textsc{Crt}(J) = \gamma$. Let α satisfying $\gamma < \alpha < \delta$ be arbitrary; then the image of α in $\pi(x)$ is ill-founded. For, there is some i such that for all $i \leq n < \omega$, $O(x|_n) \cap -U = \emptyset$. Then, $k_{0,i}(\alpha) > \gamma$, and, trivially,

$$J(J_{0,i}(\alpha)) > J(\gamma) > \gamma, \tag{6.61}$$

and

$$J(J(J_{0,i}(\alpha))) > J(J(\gamma)) > J(\gamma) > \gamma, \tag{6.62}$$

and so forth. Therefore, the image of α is ill-founded, and so $\pi(x)$ is ill-founded below δ. □

Proposition 6.7.4. *If* $X \in \Gamma$*, then* $-X \in \Gamma$.

Proof. The proof will proceed in two parts: first, we show how to build a representation $\epsilon : {}^{<\omega}\omega \to \mathcal{E}_\lambda$ such that $B_\epsilon = -X$. Then, we show how to turn ϵ into a good representation $\rho : {}^{<\omega}\omega \to \mathcal{E}_\lambda$ such that $B_\epsilon = A_\rho$.

Claim 6.7.4.1. *If* $X \in \Gamma$*, then there exists* $\epsilon : {}^{<\omega}\omega \to \mathcal{E}_\lambda$ *such that* $B_\epsilon = -X$.

Let R be some well-founded relation on λ. Then, R has the order type of some $\beta < \lambda^+$. Note that if R and R' are order isomorphic, then the image of R in $\pi(x)$ will be well-founded if and only if the image of R' is well-founded, since we can code the order isomorphism as a subset of V_λ, i.e., and element of $V_{\lambda+1}$. Therefore, without ambiguity, we can define for all $x \in \mathcal{N}$

$$\textsc{Std}(x) = \min\{\beta < \lambda^+ : \text{the image of } \beta \text{ is ill-founded in } \pi(x)\}. \tag{6.63}$$

Since λ^+ is regular, $\sup_{x \in \mathcal{N}} \textsc{Std}(x) < \lambda^+$. Therefore, let α be greater than this supremum. By Lemma 6.7.2, there therefore exists $J \in \mathcal{E}_\lambda$ such that J is iterable and $J_{0,\omega}(\eta) > \alpha$ for some $\eta < \delta$.

Now, fix $x = (x_0, x_1, \ldots) \in \mathcal{N}$. Set $j_0 = \pi(\emptyset)$, $j_1 = \pi(x_0)$, $j_2 = \pi(x_0, x_1)$, and so forth. Let $\textsc{Crt}(J') = \delta$, $J' \in \mathcal{E}_\lambda$, and inductively define, as in Section 6.6,

$$j_0^* = J' \qquad j_{m+1}^* = \big(j_{0,m+1}^*(j_m)\big)(j_m^*) \tag{6.64}$$

Then set $\epsilon(x_0, \dots, x_m) = j^*_{m+1}$. It is clear by Lemma 6.3.8 that $j^*_m \in \mathcal{E}_\lambda$ for all $m < \omega$.

We claim that $\epsilon(x)$ is strongly well-founded if and only if $\pi(x)$ is ill-founded below δ. Let ν, π_m, $\mathscr{B}_m$, and $\vec{k}_m$ also be as in Section 6.6. Note that by Lemma 6.4.8, the tower of measures given by ν induces a directed system of restricted ultrapowers:

$$\textsc{Ult}_0(V_{\lambda+1}, \nu_1) \xrightarrow{\pi_1^*} \textsc{Ult}_0(V_{\lambda+1}, \nu_2) \xrightarrow{\pi_2^*} \textsc{Ult}_0(V_{\lambda+1}, \nu_3) \xrightarrow{\pi_3^*} \cdots \tag{6.65}$$

Now, consider the following diagram:

$$\begin{array}{ccccccccc} & & \textsc{Ult}_0(V_{\lambda+1}, \nu_1) & \xrightarrow{\pi_1^*} & \textsc{Ult}_0(V_{\lambda+1}, \nu_2) & \xrightarrow{\pi_2^*} & \textsc{Ult}_0(V_{\lambda+1}, \nu_3) & \xrightarrow{\pi_3^*} & \cdots \\ & & \downarrow e_1 & & \downarrow e_2 & & \downarrow e_3 & & \\ V_{\lambda+1} & \xrightarrow{j_0^*} & V_{\lambda+1} & \xrightarrow{j_1^*} & V_{\lambda+1} & \xrightarrow{j_2^*} & V_{\lambda+1} & \xrightarrow{j_3^*} & \cdots \end{array} \tag{6.66}$$

Here, e_m is the map given by $[f] \mapsto j_{0,m}(f)(\vec{k}_m)$. Clearly this map is well-defined, and it is therefore immediately evident that Diagram 6.66 commutes. We moreover claim that e_m is the identity map. Now, it is easy to see that e_m is an elementary embedding: if $\textsc{Ult}_0(V_{\lambda+1}, \nu_m) \models \varphi(\mathbf{x})[f]$, then

$$\nu_m\{\vec{h} \in {}^m(V_{\lambda+1}) : V_{\lambda+1} \models \varphi(\mathbf{x})[\mathbf{x} = f(\vec{h})]\} = 1, \tag{6.67}$$

i.e.,

$$\vec{k}_m \in \{\vec{h} \in {}^m(V_{\lambda+1}) : V_{\lambda+1} \models \varphi(\mathbf{x})[\mathbf{x} = j^*_{0,m}(f)(\vec{h})]\} \tag{6.68}$$

and so $V_{\lambda+1} \models \varphi(\mathbf{x})[\mathbf{x} = j^*_{0,m}(f)(\vec{k}_m)]$. To see that it is surjective, simply note that

$$\textsc{Ult}_0(V_{\lambda+1}, \nu_m) \models \text{“There exists } \lambda \text{ such that for all } x,\ x \subset V_\lambda\text{”} \tag{6.69}$$

and, moreover, $\lambda \in \textsc{Ult}_0(V_{\lambda+1}, \nu_m)$, as witnessed by the elementary embedding $\iota_{\nu_m} : V_{\lambda+1} \to \textsc{Ult}_0(V_{\lambda+1}, \nu_m)$ (cf. Corollary 6.4.5). Hence, it follows that

$$(\lambda)^{\textsc{Ult}_0(V_{\lambda+1}, \nu_m)} \geq (\lambda)^{V_{\lambda+1}}. \tag{6.70}$$

where λ is defined as in Sentence 6.8.2. However, since there cannot be an elementary embedding from a transitive set M_1 to a transitive set M_2 if $\textsc{Ord}^{M_2}$ is an initial segment of $\textsc{Ord}^{M_1}$, it follows that $\textsc{Ult}_0(V_{\lambda+1}, \nu_m) = V_{\lambda+1}$, so e_m is in fact the identity map.

Recall that by Lemma 6.4.8, the direct limit of the rows of Diagram 6.66 are strongly well-founded if and only if ν is countably complete. By Lemma 6.6.2, ν is countably complete if and only if $\pi(x)$ is ill-founded below δ. Hence, $\epsilon(x)$, which is the direct limit of the sequence of embeddings

$$V_{\lambda+1} \xrightarrow{j_0^*} V_{\lambda+1} \xrightarrow{j_1^*} V_{\lambda+1} \xrightarrow{j_2^*} \cdots \tag{6.71}$$

is strongly well-founded if and only if $\pi(x)$ is ill-founded below δ. Therefore $B_\epsilon = -A_\pi$.

Claim 6.7.4.2. *If $X = B_\epsilon$ for some $\epsilon : {}^{<\omega}\omega \to \mathcal{E}_\lambda$, then there exists a good representation $\rho : {}^{<\omega}\omega \to \mathcal{E}_\lambda$ such that $A_\rho = B_\epsilon$.*

Keeping the same notation as in the previous claim, recall that $J_{0,\omega}(\eta) > \alpha$ for some $\eta < \delta$. Suppose $x \in \mathcal{N}$ is such that $\epsilon(x)$ is not strongly well-founded; then recalling that $\alpha > \textsc{Std}(x)$, note that $j_{0,\omega}(\beta)$ is ill-founded for any $\beta > \alpha$. Hence, $j_{0,\omega}(J_{0,\omega}(\eta))$ is ill-founded, and, moreover $\eta < \delta$. Therefore, $(j_{0,\omega} \circ J_{0,\omega})$ is an elementary embedding of $V_{\lambda+1}$ into a model ill-founded below δ whenever $j_{0,\omega}$ is ill-founded; conversely, since J is iterable, it follows that $(j_{0,\omega} \circ J_{0,\omega})$ is an elementary embedding of $V_{\lambda+1}$ into a strongly well-founded model whenever $j_{0,\omega}$ is strongly well-founded. Thus, our goal is to produce a sequence of embeddings for which the canonical map into the direct limit is given by $j_{0,\omega} \circ J_{0,\omega}$.

To that end, consider the following diagram, where N_J denotes $\varinjlim(V_{\lambda+1}, J_n)$, and N_{j^*} denotes $\varinjlim(V_{\lambda+1}, j^*_n)$.

$$
\begin{array}{ccccccccc}
V_{\lambda+1} & \xrightarrow{J_0} & V_{\lambda+1} & \xrightarrow{J_1} & V_{\lambda+1} & \xrightarrow{J_2} & \cdots & & N_k \\
\downarrow{\scriptstyle j^*_0} & & \downarrow{\scriptstyle j^*_0} & & \downarrow{\scriptstyle j^*_0} & & & & \downarrow{\scriptstyle j^*_0} \\
V_{\lambda+1} & \xrightarrow{j^*_0(J_0)} & V_{\lambda+1} & \xrightarrow{j^*_0(J_1)} & V_{\lambda+1} & \xrightarrow{j^*_0(J_2)} & \cdots & & N_k \\
\downarrow{\scriptstyle j^*_1} & & \downarrow{\scriptstyle j^*_1} & & \downarrow{\scriptstyle j^*_1} & & & & \downarrow{\scriptstyle j^*_1} \\
V_{\lambda+1} & \xrightarrow{j^*_{0,2}(J_0)} & V_{\lambda+1} & \xrightarrow{j^*_{0,2}(J_1)} & V_{\lambda+1} & \xrightarrow{j^*_{0,2}(J_2)} & \cdots & & N_k \\
\downarrow{\scriptstyle j^*_2} & & \downarrow{\scriptstyle j^*_2} & & \downarrow{\scriptstyle j^*_2} & & & & \downarrow{\scriptstyle j^*_2} \\
\vdots & & \vdots & & \vdots & & & & \vdots \\
N'_{j^*} & \xrightarrow{j_{0,\omega}(k_0)} & N'_{j^*} & \xrightarrow{j_{0,\omega}(k_1)} & N'_{j^*} & \xrightarrow{j_{0,\omega}(k_2)} & \cdots & & M
\end{array}
\tag{6.72}
$$

The vertical maps along the right-most column and the bottom-most row are induced by the universal property of direct limits. Indeed, a straightforward diagram chase on the body of the diagram yields that the direct limits of the right-most row and the bottom-most column are isomorphic. We let M denote this limit. As the right column witness, $j_{0,\omega} \circ J_{0,\omega}$ is an elementary embedding of $V_{\lambda+1}$ into M.

We seek to construct a sequence of λ-embeddings h_0, h_1, h_2, $\ldots$ such that the limit of

$$V_{\lambda+1} \xrightarrow{h_0} V_{\lambda+1} \xrightarrow{h_1} V_{\lambda+1} \xrightarrow{h_2} \cdots \tag{6.73}$$

is M. We also need the sequence h_0, h_1, $\ldots$ to satisfy Condition (i) in Definition 6.5.1. With this in mind, note that there exists n_0 such that $J_{0,n_0}(\eta) > \textsc{Crt}(j^*_0)$, since $J_{0,\omega}(\eta) \geq \lambda$. Set $h_0 = J_{0,n_0}$. Then, note that

$$\textsc{Crt}(h_0) = \min_{0 \leq m < n_0} \textsc{Crt}(J_n) \leq \textsc{Crt}(J) < \delta. \tag{6.74}$$

Next, set $h_1 = j^*_0$. Then $h_0(\eta) > \textsc{Crt}(h_1)$ by construction. Since $\big(j^*_0(J_{n_0,\omega})\big)(j^*_0(\eta))$ is greater than or equal to λ, there exists n_1 such that $\big(j^*_0(J_{n_0,n_1})\big)(j^*_0(\eta))$ is greater than $\textsc{Crt}(j_2)$. Set $h_2 = j^*_0(J_{n_0,n_1})$, and note that

$$\textsc{Crt}(h_2) \leq \textsc{Crt}(j^*_0(J_{0,n_0})) \leq j^*_0(J^*_{0,n_0}(\eta)) \tag{6.75}$$

since $\eta > \textsc{Crt}(J)$ and $\textsc{Crt}(J_n) = J_{0,n}(\textsc{Crt}(J))$. Then, set $h_3 = j^*_1$, and so on.

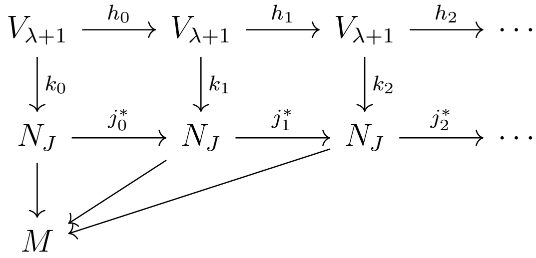
Then, by construction, $h_{2m}(\eta)$ will be greater than $\textsc{Crt}(h_{2m+1})$,

$$\textsc{Crt}(h_{2m}) = \min_{n_m \leq i < n_{m+1}} j^*_{0,m}(\textsc{Crt}(J_i)) = j^*_{0,m}(\textsc{Crt}(J_{n_m})) < j^*_{0,m}(J_{0,n_m}(\eta)) = h_{0,2m-1}(\eta). \quad (6.76)$$

where the final inequality follows by the commutativity of Diagram 6.72.

To see that $\varinjlim(V_{\lambda+1}, h_m) = M$, note that it suffices by the universal property of direct limits to accomplish the following:

(i) Find maps $\langle k_i : i < \omega \rangle$ such that the following diagram commutes

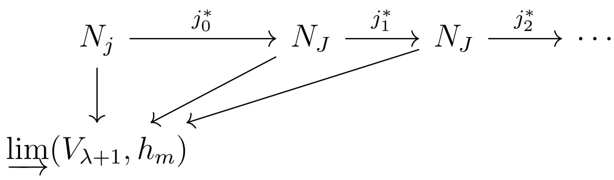


(6.77)

(ii) Find, for all $m < \omega$, maps $\langle k^m_i : i < \omega \rangle$ such that the following diagram commutes

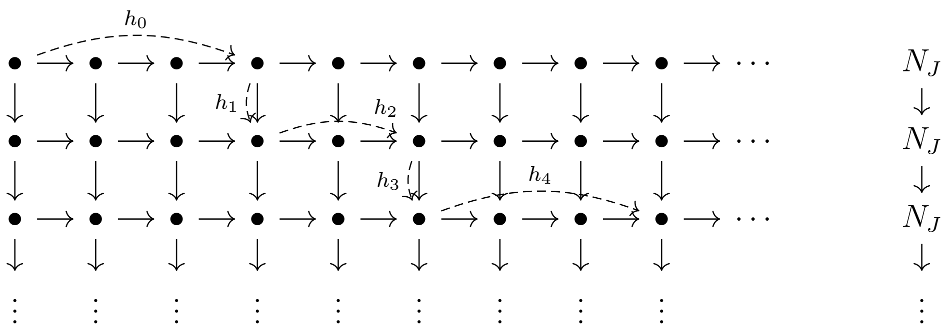
$$\begin{array}{ccccccc}
V_{\lambda+1} & \xrightarrow{j^*_{0,m}(J_0)} & V_{\lambda+1} & \xrightarrow{j^*_{0,m}(J_1)} & V_{\lambda+1} & \xrightarrow{j^*_{0,m}(J_2)} & \cdots \\
\downarrow k^m_0 & & \downarrow k^m_1 & & \downarrow k^m_2 & & \\
V_{\lambda+1} & \xrightarrow{h_0} & V_{\lambda+1} & \xrightarrow{h_1} & V_{\lambda+1} & \xrightarrow{h_2} & \cdots
\end{array} \quad (6.78)$$

as this induces the commutative diagram



(6.79)

However, note that, letting the solid vertical and horizontal arrows be as in Diagram 6.72, the h_0, h_1, h_2 proceeds through Diagram 6.72, for example, in the following manner



(6.80)

Hence the k_i's and the k^m_i can be found with a simple diagram chase.

To conclude, note that $h_m \in \mathcal{E}_\lambda$. Hence, setting $\rho(x{\restriction}_m) = h_m$ gives the desired representation. □

Definition 6.7.5 (C-representations)**.** Recall our convention that ${}^{<\omega}\omega \times {}^{<\omega}\omega$ denotes the the set of pairs of sequences of natural numbers *of equal length.* Then, let $\pi : {}^{<\omega}\omega \times {}^{<\omega}\omega \to \mathcal{E}_0$ fulfill two conditions: for any fixed $x, y \in \mathcal{N}$, either $\pi(x, y)$ is strongly well-founded, or it is ill-founded below δ.

In this case, we say that the set

$$C_\pi = \{x \in \mathcal{N} : \pi(x, y) \text{ is ill-founded below } \delta \text{ for all } y \in \mathcal{N}\} \tag{6.81}$$

has a *C-representation.*

The use of C-representations will allow us to prove the following proposition.

Proposition 6.7.6. *The point class Γ is closed under projections and countable unions.*

To prove Proposition 6.7.6, we shall make use of the following lemma.

Proposition 6.7.7. *A set $X \subset \mathcal{N}$ has an A-representation if and only if it has a C-representation.*

Proof. One direction is trivial: if $X = A_\pi$, then set $\pi' : {}^{<\omega}\omega \times {}^{<\omega}\omega \to \mathcal{E}_0$ by $\pi'(x, y) = \pi(x)$. It follows immediately that $C_{\pi'} = A_\pi$.

Next, suppose $X = C_\pi$. We shall prove in a manner exactly analogous to Proposition 6.7.4 that X has a B-representation. Then, it follows by Claim 6.7.4.2 that X has an A-representation.

The construction is almost exactly parallel to that in Proposition 6.7.4; however, because of the added complexity of C-representations, the details are superficially more complicated.

Recall $\mathfrak{s} : \omega \to {}^{<\omega}\omega$, the enumeration of the finite sequences with the property that if t extends s, t and s both being finite sequences, then $\mathfrak{s}_i = s$ and $\mathfrak{s}_n = t$ for $i < n$. Let X_i again denote the set ${}^{i}(\mathcal{E}_0)$, and let $\pi_i : X_{i+1} \to Y_i$ denote the canonical projection onto the first i coordinates. We seek to define a tower of measures on $\langle X_i : i < \omega \rangle$.

Fix $x \in \mathcal{N}$, and note that $\pi_x : {}^{<\omega}\omega \to \mathcal{E}_\lambda$ given by $\pi_x(s) = \pi(x{\restriction}_{|s|}, s)$ is a good representation for every $x \in \mathcal{N}$. For simplicity, let $j^s_m = \pi_x(s{\restriction}_m)$, $j^s_{0,n} = \pi_x(s{\restriction}_n) \circ \pi_x(s{\restriction}_{n-1}) \circ \cdots \circ \pi_x(s{\restriction}_0)$, and so on, as is consistent with our earlier notation. We abbreviate $j^s_{|s|}$ by j^s. Likewise, for $y \in \mathcal{N}$, we write j^y_0, j^y_1, j^y_2, etc. for the embeddings $\pi_x(y{\restriction}_0)$, $\pi_x(y{\restriction}_1)$, $\pi_x(y{\restriction}_2)$, and so on.

Then, exactly in parallel to Section 6.6.2, we can define ν, a tower of measures, in the following manner:

(i) The measure ν_m has generic point $(k^m_0, \ldots, k^m_{m-1})$, inductively defined as follows:

  (a) The embedding k^1_0 is J and $j^*_0 = J$, so that $\nu_1(S) = 1$ if and only if $k^1_0 \in j^*_{0,1}(S)$, as is easily checked;

  (b) If $(k^{m-1}_0, \ldots, k^{m-1}_{m-1})$ is the generic point for ν_m, then $k^m_{m-1} = j^*_{m-1}$; furthermore, if $\mathfrak{s}_n = \mathfrak{s}_{m+1}{\restriction}_{|\mathfrak{s}_{m+1}|-1}$—i.e., $\mathfrak{s}_n$ is the one-term truncation of $\mathfrak{s}_{m+1}$—then it follows that $j^*_m = \big(j_{0,m}(j^{s'})\big)(k^m_n)$ and the generic point for ν_{m+1} is $(j^*_m(k^{m-1}_0), \ldots, j^*_m(k^{m-1}_{m-1}), j^*_m)$;

(ii) If, exactly as in Proposition 6.7.4, we set $\epsilon(x\restriction_m) = j^*_m$, then the diagram

$$\begin{array}{ccccccccc} & & \textsc{Ult}_0(V_{\lambda+1},\nu_1) & \overset{\pi_1^*}{\to} & \textsc{Ult}_0(V_{\lambda+1},\nu_2) & \overset{\pi_2^*}{\to} & \textsc{Ult}_0(V_{\lambda+1},\nu_3) & \overset{\pi_3^*}{\to} & \cdots \\ & & \downarrow{\scriptstyle \text{EV}_1} & & \downarrow{\scriptstyle \text{EV}_2} & & \downarrow{\scriptstyle \text{EV}_3} & & \\ V_{\lambda+1} & \xrightarrow{j_0^*} & V_{\lambda+1} & \xrightarrow{j_1^*} & V_{\lambda+1} & \xrightarrow{j_2^*} & V_{\lambda+1} & \xrightarrow{j_3^*} & \cdots \end{array} \tag{6.82}$$

commutes, and the vertical arrows are the identity;

(iii) A ν-generic m-tuple $\vec{k} = (k_0, k_1, \ldots, k_m)$, satisfies that the critical points of $\pi^y(\vec{k})$ form the initial segment of a β-sequence for $\pi_x(y)$, i.e.,

$$\begin{aligned} \textsc{Crt}(k_{i_0}) < \delta \quad j_0^y(\textsc{Crt}(k_{i_0})) > \textsc{Crt}(k_{i_1}) \quad \ldots \\ \ldots \quad j_{n-1}^y(\textsc{Crt}(k_{i_{n-1}})) > \textsc{Crt}(k_{i_n}) \end{aligned} \tag{6.83}$$

(Heuristically, the only difference between this construction and that of Proposition 6.7.4 is that the latter was carried out only for a single branch of ${}^{<\omega}\omega$, i.e., a single $y \in \mathcal{N}$; here, the construction is carried out for *every* branch of ${}^{<\omega}\omega$ simultaneously. For this reason, Points (i) and (ii) follow for precisely the same reasons as they did before, and a slightly more careful version of the argument in Lemma 6.6.1 yields Point (iii).)

Therefore, it follows that $\epsilon(x)$ is strongly well-founded if and only if ν is countably complete; again by Lemma 6.6.2, the tower ν is countably complete if and only if $\pi_x(y)$ is ill-founded below δ for every $y \in \mathcal{N}$. Therefore, it follows that $B_\epsilon = C_\pi$, as desired. □

Corollary 6.7.8. *The point class* Γ *is closed under projection and countable unions.*

Proof. This is merely a direct application of Propositions 6.7.4 and 6.7.7. Let $f : {}^{<\omega}\omega \times {}^{<\omega}\omega \to {}^{<\omega}\omega$ be a bijection, with the property that if (s,t) extends (s',t'), then $f(s,t)$ extends $f(s',t')$. Then, f induces a homeomorphism $f^* : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$. Let A_π be give; then, we seek $\epsilon : {}^{<\omega}\omega \times {}^{<\omega}\omega \to \mathcal{E}_\lambda$ such that $\epsilon(x,y)$ is strongly well-founded or ill-founded below delta exactly as $\pi(f(x,y))$ is. Let $\rho : {}^{<\omega}\omega \to \mathcal{E}_\lambda$ be such that $A_\rho = -A_\pi$.

However, ϵ is not hard to construct. Let s and t be one-term extensions of s' and t', respectively. Then, set $p' = f(s',t')$ and $p = f(s,t)$. Let $p_0 = p$, $p_1 = p'\restriction_{|p|+1}$, $\ldots$, $p_n = p'$. Then, set

$$\epsilon(s,t) = \rho(p_n) \circ \cdots \circ \rho(p_1). \tag{6.84}$$

It follows as an immediate consequence that

$$\{(x,y) \in \mathcal{N} \times \mathcal{N} : \epsilon(x,y) \text{ is ill-founded below } \delta\} = (f^*)^{-1}(-A_\pi). \tag{6.85}$$

Therefore, there exists $\rho' : {}^{<\omega}\omega$ such that $A_{\rho'} = -C_\epsilon$. But this is exactly the desired projection of $(f^*)^{-1}(A_\pi)$.

Next, let A_{π_0}, A_{π_1}, A_{π_2}, and so on all be in Γ. Choose $y_0 \neq y_1 \neq y_2$ all in $\mathcal{N}$, and let $A_{\pi_0} = -A_{\rho_0}$, $A_{\pi_1} = -A_{\rho_1}$, and so on. Choose $k \in \mathcal{E}_\lambda$, so that k is iterable. Then, set

$$\epsilon(s,t) = \begin{cases} k_{|s|} & s \not\subset y_n \text{ for any } n \\ \rho_n(t) & t \subset y_n \end{cases} \tag{6.86}$$

Then $-C_\epsilon = \bigcap_{n<\omega} A_{\pi_n}$, so that again by Propositions 6.7.4 and 6.7.7, $\bigcap_{n<\omega} A_{\pi_n} \in \Gamma$. □

Therefore, Γ contains the projective sets. The final piece required to prove projective determinacy is the theorem below.

Theorem 6.7.9. *The point class* Γ *is determined.*

Proof. Let the set A_π be given. We consider two games:

$$\begin{array}{c|c} \multicolumn{2}{c}{G_1(\pi)} \\ \text{I} & \text{II} \\ \hline n_0 & \\ & m_0 \\ n_1 & \\ & m_1 \\ n_2 & \\ & m_2 \\ \vdots & \vdots \end{array} \qquad \begin{array}{c|c} \multicolumn{2}{c}{G_2(\pi)} \\ \text{I} & \text{II} \\ \hline (\beta_0, n_0) & \\ & m_0 \\ (\beta_1, n_1) & \\ & m_1 \\ (\beta_2, n_2) & \\ & m_2 \\ \vdots & \vdots \end{array} \tag{6.87}$$

The game $G_1(\pi)$ is simply the standard game $G_{A_\pi}(\mathcal{N})$. The game $G_2(\pi)$ requires that Player I play an ordinal $0 \leq \beta_i < \lambda$ on her i-th turn. Player I wins if and only if $\pi(x * y)$, where $x = (n_0, n_1, \ldots)$ and $y = (m_0, m_1, \ldots)$, is ill-founded below δ and $\langle \beta_i : i < \omega \rangle$ is a β-sequence witnessing its ill-foundation.

Notice that $G_2(\pi)$ is a closed game for Player I: if she loses, it must be because she played an ordinal at some finite turn which failed to continue the β-sequence she had been building. Therefore, exactly as in Theorems 5.4.14 and 5.4.8, $G_2(\pi)$ is determined, by Lemma 5.2.3.

Now, it is clear that if Player I has a winning strategy for $G_2(\pi)$, then she has a winning strategy for $G_1(\pi)$.

Suppose Player II has a winning strategy σ for $G_2(\pi)$; then, we must produce a strategy σ' for Player II in $G_1(\pi)$.

First, suppose in $G_1(\pi)$, Player I begins by playing n_0. Let J' be the fixed λ-embedding with critical point δ, as in Lemma 6.7.2. Then, we can define the measure ν_1 on singletons. Let

$$S_0^\ell = \{k \in \mathcal{E}_0 : \sigma\big((\textsc{Crt}(k), n_0)\big) = \ell\} \tag{6.88}$$

and notice that $S_0^\ell \in L_{\omega_1}(V_{\lambda+1})$ for all $\ell < \omega$. Therefore, ν_1 can measure S_0^ℓ. Moreover, since $\bigcup_{\ell<\omega} S_0^\ell = \mathcal{E}_{\beta_0}$, there is exactly one ℓ_0 such that $\mu_J^{\beta_0}(S_0^{m_0}) = 1$. Then, set $\sigma'(n_0) = \ell_0$.

Now, proceeding by induction, suppose σ' has been defined for Player II's first i moves in $G_1(\pi)$, and on her $i+1$-st turn, Player I responds by n_i. Now, we can use the sequence $j_0 = \pi(\emptyset)$, $j_1 = \pi(n_0)$, $j_2 = \pi(n_0, m_0)$ to construct the measures ν_1, ν_1, $\ldots$, ν_{i+1}, as in Section 6.6. Then, define

$$S_i^\ell = \big\{(k_0, \ldots, k_i) \in X_i : \sigma\big(\langle \textsc{Crt}(k_0), n_0 \rangle, m_0, \ldots \langle \textsc{Crt}(k_i), n_i \rangle\big) = \ell\big\} \tag{6.89}$$

and note that for exactly one fixed $\ell_0 < \omega$, $\nu_{i+1}(S_i^{\ell_0}) = 1$. Set $\sigma'(n_0, m_0, \ldots, n_i) = \ell_0$.

Before we proceed, it remains to check a consistency requirement, namely, that $\pi_{i,\ell}``S_i^{m_i} = S_\ell^{m_\ell}$, i.e., that for $(k_0, \ldots, k_{i-1}) \in S_i^{m_i}$ that $\sigma(\textsc{Crt}(k_0), n_0)$ actually equals m_0, that $\sigma\big((\textsc{Crt}(k_0), n_0), m_0, (\textsc{Crt}(k_1), n_1)\big) = m_1$, and so forth. However, this follows by induction once we note that $\mu_M(S_\ell^{\ell'}) = 0$ for $\ell' \neq m_\ell$ and that $\mu_M(\pi_{i,\ell}``S_i^{m_i}) = 1$.

Then, we claim that σ' is a winning strategy for Player II. For, suppose not. Then $\pi(x * y)$ is ill-founded below δ. Let $\langle \beta_i' : i < \omega\rangle$ be a β-sequence witnessing its ill-foundation. Then ν is a countably complete tower of measures. Moreover, since $\nu_i(S_i^{m_i}) = 1$ for all $i < \omega$, so there is some thread $\langle \mathfrak{h}_i : i < \omega\rangle$. Since a ν_m-generic sequence of embeddings have critical points forming the first m elements of a β-sequence, it follows that we can assume that for the sequence

$$\bigcup_{i<\omega} \mathfrak{h}_i = (h_0, h_1, \dots) \tag{6.90}$$

the sequence $\langle \textsc{Crt}(h_i) : i < \omega\rangle$ is a β-sequence. Therefore,

$$p = \big(\langle \textsc{Crt}(h_0), n_0\rangle, m_0, \langle \textsc{Crt}(h_1), n_1\rangle, \dots\big) \tag{6.91}$$

is a losing play for Player II. However, by the consistency requirement, if we set

$$p_{\mathrm{I}} = \big(\langle \textsc{Crt}(h_0), n_0\rangle, \langle \textsc{Crt}(h_1), n_1\rangle, \dots\big) \tag{6.92}$$

then $y = \sigma(p_{\mathrm{I}})$, and hence $p_{\mathrm{I}} * \sigma(p_{\mathrm{I}})$ is a losing play for Player II, which is impossible, since σ was assumed a winning strategy.

Therefore, σ' is a winning strategy for Player II. Thus, $G_1(\pi)$ is determined, as desired. □

6.8 Extension to $\mathbf{AD}_{L(\mathbb{R})}$

As evidenced by Theorem 6.7.1, A-representations provide a very powerful canonical form for sets in the point class Γ. Indeed, Γ enjoys significantly stronger closure properties than were ultimately required in Section 6.7. In this section, we investigate the extent of these closure properties by giving a sketch of the proof that $\mathbf{AD}_{L(\mathbb{R})}$ is a consequence of $\mathbf{I}_0$.

Note that we assume familiarity with forcing.

Theorem 6.8.1. *Assume $\mathbf{I}_0$. Then $\mathbf{AD}_{L(\mathbb{R})}$ holds.*

We need a number of background lemmata in order to prove Theorem 6.8.1.

Lemma 6.8.2. *Assume $\mathbf{I}_0$, and let $\mathfrak{j} : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ be a witness. Then there is a club set $C \subset \lambda$ with the following properties:*

1. *The embedding $\mathfrak{j}$ maps C to itself, i.e., $\mathfrak{j}(C) = C$;*
2. *For all $\bar\lambda \in C$ of cofinality ω, the following property holds:*
   (a) *There exists a fixed $\delta < \bar\lambda$ such that for any $\alpha < \bar\lambda^+$, there is a $\bar\lambda$-embedding[4] $J : V_{\gamma+1} \to V_{\gamma+1}$ such that $J_{0,\omega}(\eta) > \alpha$ for some $\eta < \delta$ and a $\bar\lambda$-embedding J' with critical point δ;*

[4]Here, by "$\bar\lambda$-embedding," we mean an elementary embedding $k : V_{\bar\lambda+1} \to V_{\bar\lambda+1}$ extending to an elementary embedding from $L_{\bar\lambda}(V_{\bar\lambda+1})$ to $L_{\bar\lambda}(V_{\bar\lambda+1})$.

(b) There exists an elementary embedding $k : L_{\bar\lambda+\bar\lambda}(V_{\bar\lambda+1})$.

Proof. This follows from Theorem 6.10 of [Cra15] and Corollary 149 of [Woo11c].[5] □

Lemma 6.8.3. *Suppose* Γ' *is a* σ*-algebra on* $\mathcal{N}$ *closed under continuous images and containing the open sets, and moreover that* Γ' *is determined. Then, either* $\Gamma' \subset L(\mathbb{R})$*, or* $L(\mathbb{R}) \cap \mathcal{P}(\mathbb{R}) \subset \Gamma'$.

Proof. The proof is a standard argument using Wadge games. □

Remark 6.8.4. The naïve path toward proving Theorem 6.8.1 was to strengthen Proposition 6.7.7. However, when this proved impossible, Lemma 6.8.3 allowed an end-run around the difficulty of constructing explicit A-representations for sets in $L(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$.

Lemma 6.8.5. *Suppose that* γ *is a Woodin cardinal that is a limit of Woodin cardinals, with a measurable cardinal above* γ*. Then, no forcing construction in* V_γ *can change the theory of* $L(\mathbb{R})$ *with real parameters.*

Proof. See Corollary 3.1.16 in [Lar04]. □

Our strategy to prove Theorem 6.8.1 will be the following: we shall create a very complicated subset of V_{λ_0+1} for some $\lambda_0 \in C$ of cofinality ω, which, upon collapsing V_λ to ω, yields a set of reals in $V[G]$. In particular, every set of reals in $L(\mathbb{R})^{V[G]}$ arises from a subset of V_{λ_0+1} of a particularly simple form. Therefore, our original subset is not in $L(\mathbb{R})^{V[G]}$, and hence Lemma 6.8.3 obtains in $V[G]$, and so **AD** holds in $L(\mathbb{R})^{V[G]}$. By Lemma 6.8.5, it therefore follows that **AD** holds in $L(\mathbb{R})^V$.

Definition 6.8.6 (Σ_n-Embeddings)**.** Given two models $\mathcal{M}_0 = (M_0, \ldots)$ and $\mathcal{M}_1 = (M_1, \ldots)$ in the same language $\mathcal{L}$, we say that $j : M_0 \to M_1$ is a Σ_n*-embedding* if and only if, for every Σ_n formula $\varphi(\vec{\mathbf{x}}) \in \textsc{Form}_{\mathcal{L}}$ and parameters $\vec{m} \in M_0$,

$$\mathcal{M}_0 \models \varphi(\mathbf{x})[\vec{m}] \iff \mathcal{M}_1 \models \varphi(\mathbf{x})[j(\vec{m})]. \tag{6.93}$$

Remark 6.8.7. As in Lemma 6.3.8, notice that an embedding $j : M_0 \to M_1$ is elementary if and only if it is a Σ_n-embedding for every n.

Definition 6.8.8. Let $j : V_{\bar\lambda} \to V_{\bar\lambda}$ be an elementary embedding for $\bar\lambda \in C$. Note that, as in Proposition 6.3.2, j has an extension to an embedding $j^+ : V_{\bar\lambda+1} \to V_{\bar\lambda+1}$. By a slight abuse of notation, we say that j is a Σ_n*-embedding* if and only if j^+ is a Σ_n-embedding in the sense of Definition 6.8.6, i.e., for all Σ_n formulas $\varphi(\mathbf{x})$,

$$V_{\bar\lambda+1} \models \varphi(\mathbf{x})[a] \iff V_{\bar\lambda+1} \models \varphi(\mathbf{x})[j^+(a)]. \tag{6.94}$$

Lemma 6.8.9. *If* $j : V_{\bar\lambda} \to V_{\bar\lambda}$ *is a* Σ_1 *embedding, then* j *is a* Σ_2 *embedding. More generally, if* $j : V_{\bar\lambda} \to V_{\bar\lambda}$ *is a* Σ_{2n+1} *embedding, then* j *is a* Σ_{2n+2} *embedding.*

[5] Note that as stated, there is a slight technical omission: the immediate consequence of these results is the existence of Σ_0-embeddings from $L_{\bar\lambda}(V_{\bar\lambda+1})$ to $L_{\bar\lambda}(V_{\bar\lambda+1})$, rather than elementary embeddings. An easy solution is to note that, in fact, the entire content of Sections 6.2 through 6.7 still follows after replacing $\mathcal{E}_\lambda$ with embeddings in $\mathcal{E}_0$ extending to Σ_0 embeddings of $L_\lambda(V_{\lambda+1})$ to $L_\lambda(V_{\lambda+1})$. See Definition 6.8.6 for the precise formulation of a Σ_0 embedding.

Proof. See Theorem 2 in [Lav01]. □

Definition 6.8.10. For $\bar{\lambda} \in C$ of cofinality ω, let $I(\bar{\lambda})$ denote the set

$$\{k\restriction_{V_{\alpha_n}} : k \text{ is an elementary embedding } V_{\bar{\lambda}} \to V_{\bar{\lambda}}\} \tag{6.95}$$

where $\langle \alpha_n : n < \omega \rangle$ is a fixed cofinal sequence in $\bar{\lambda}$.

Let $\mathcal{E}_{\bar{\lambda}}$ now denote the set of $\bar{\lambda}$-embeddings.[6] Then, we further define the following two sets:

$$\mathcal{E}^0_{I(\bar{\lambda})} = \{k \in \mathcal{E}_{\bar{\lambda}} : k_n\restriction_{V_{\alpha_m}} \in I(\bar{\lambda}) \text{ for all } n, m < \omega\} \tag{6.96}$$

and

$$\mathcal{E}_{I(\bar{\lambda})} = \{k \in \mathcal{E}^0_{I(\bar{\lambda})} : k \text{ is a } \Sigma_1\text{-embedding}\}. \tag{6.97}$$

We shall also denote these sets by I, $\mathcal{E}_I$, and $\mathcal{E}^0_I$ when $\bar{\lambda}$ is clear from context.

Lemma 6.8.11. *The set $\mathcal{E}_I$ is not Σ_2-definable with parameters in $V_{\bar{\lambda}+1}$.*

Proof. Suppose there were some Σ_2 formula $\varphi(\mathbf{x}, \mathbf{y})$ and some parameter $p \in V_{\bar{\lambda}+1}$ such that for all $k : V_{\bar{\lambda}} \to V_{\bar{\lambda}}$, $k \in \mathcal{E}_I \iff \varphi(k, p)$. Then, note that the set

$$E = \{(k, a) : k \in \mathcal{E}_I \text{ and } k(a) = p\} \tag{6.98}$$

is not empty. For, being a Σ_1-embedding is a Σ_2 property, and so, by Equation 6.97 and Lemma 6.8.9, if $k \in \mathcal{E}_I$, then $k(k) \in \mathcal{E}_I$. Let $j_0 : V_{\bar{\lambda}+\bar{\lambda}}(V_{\bar{\lambda}+1}) \to V_{\bar{\lambda}+\bar{\lambda}}(V_{\bar{\lambda}+1})$, as in Lemma 6.8.2. Then, the restriction of j_0 to $V_{\bar{\lambda}+1}$ is trivially Σ_3, so $j_0(\mathcal{E}_I) = \mathcal{E}_I$. Therefore,

$$(j_0, p) \in j_0(E) = \{(k, a) : k \in \mathcal{E}_I \text{ and } k(a) = j_0(p)\} \tag{6.99}$$

and so, by elementariness, E is non-empty.

Now, let $(k, a) \in E$. Suppose that $\varphi(h, a)$ and $h \in \mathcal{E}^0_I$. We claim that $h \in \mathcal{E}_I$. For, suppose not; then let b witness that h is not a Σ_1-embedding. Then $k(b)$ will witness that $k(h)$ is not a Σ_1-embedding; however, $\varphi(k(h), k(a))$ holds, so in fact $k(h) \in \mathcal{E}_I$, contrary to assumption.

Now, let h be of minimal critical point such that $(h, a) \in E$ for some a. Note that since

$$V_{\bar{\lambda}+1} \models \text{"There exists } j \text{ such that } j(j) = h(h),\ j \in \mathcal{E}^0_I, \text{ and } \varphi(j, h(a))\text{"} \tag{6.100}$$

it follows that

$$V_{\bar{\lambda}+1} \models \text{"There exists } j \text{ such that } j(j) = h,\ j \in \mathcal{E}^0_I, \text{ and } \varphi(j, a)\text{".} \tag{6.101}$$

Let h_0 witness this; then $h_0 \in \mathcal{E}^0_I$ and $\varphi(h_0, a)$, so $h_0 \in \mathcal{E}_I$. However, this is a contradiction, since $h_0(h_0(a)) = p$ and $\mathrm{CRT}(h_0) < \mathrm{CRT}(h)$. □

Remark 6.8.12. Our task now is to build an "A-representation" for $\mathcal{E}^0_I - \mathcal{E}_I$. Let $\lambda_0 \in C$ be greater than $\mathrm{CRT}(j_0)$, and let $\alpha^0_0 < \alpha^0_1 < \alpha^0_2 < \ldots$ witness the fact that $\mathrm{COF}(\lambda_0) = \omega$. Then, we set $\lambda_1 = \mathfrak{j}(\lambda_0)$, $\alpha^1_0 = \mathfrak{j}(\alpha^0_0)$, $\alpha^1_1 = \mathfrak{j}(\alpha^0_1)$, and so on.

[6] That is, $\mathcal{E}_{\bar{\lambda}}$ is the set of $\bar{\lambda}$-embeddings, not embeddings $j : V_\lambda \to V_\lambda$ extending to an embedding $L_{\bar{\lambda}}(V_{\lambda+1})$ to $L_{\bar{\lambda}}(V_{\lambda+1})$. This is in opposition to the notation maintained in Sections 6.3 to 6.7.

Let us abbreviate $k\restriction_{V_\beta}$ by k^β. Then, set $\textsc{Seq}(\lambda_0)$ to be the set of finite sequences of the form

$$\left\langle k^{\alpha^0_{m_0}}, k^{\alpha^0_{m_1}}, \ldots, k^{\alpha^0_{m_i}} \right\rangle \tag{6.102}$$

where $\langle m_n : n < \omega \rangle$ is an increasing sequence of indices, each element of the sequence is an element of $I(\lambda_0)$, and furthermore

$$k^{\alpha^0_{m_0}} \subset k^{\alpha^0_{m_1}} \subset \ldots \subset k^{\alpha^0_{m_i}}. \tag{6.103}$$

Let $\textsc{Seq}(\lambda_1)$ be defined analogously. We denote by $\overline{\textsc{Seq}}(\lambda_0)$ to be those $f \in {}^\omega(V_{\lambda_0+1})$ such that $f\restriction_n \in \textsc{Seq}(\lambda_0)$ for all $n < \omega$.

Then a *good* representation π is a map $\pi : \textsc{Seq}(\lambda_0) \to \mathcal{E}_{\lambda_1}$ satisfying, *mutatis mutandis*, the requirements Definition 6.5.1,[7] and A_π is defined accordingly as

$$\left\{ x = \left\langle k^{\alpha^0_{m_0}}, k^{\alpha^0_{m_1}}, \ldots \right\rangle \in \overline{\textsc{Seq}}(\lambda_0) : \pi(x) \text{ is ill-founded below } \delta_1 \right\}. \tag{6.104}$$

We need the following characterization of Σ_1 embeddings.

Lemma 6.8.13. *An embedding $k : V_\lambda \to V_\lambda$ extends to an embedding $k^+ : V_{\lambda+1} \to V_{\lambda+1}$ that is Σ_1 if and only if $k^+_{0,\omega}$ is strongly well-founded.*

Proof. This is a direct consequence of Theorem 1.3 in [Lav01] and Theorem 6.2.2. □

We now seek to construct a certain $\pi : \textsc{Seq}(\lambda_0) \to \mathcal{E}_{\lambda_1}$, which will be related to $\mathcal{E}^0_{I(\lambda_0)} - \mathcal{E}_{I(\lambda_0)}$ in a simple way. Because A-representations are for subsets of $\overline{\textsc{Seq}}(\lambda_0)$ rather than subsets of $\mathcal{E}_{\lambda_0}$, we cannot directly represent $\mathcal{E}^0_{I(\lambda_0)} - \mathcal{E}_{I(\lambda_0)}$.

The important observation is that if $k \in \mathcal{E}^0_I$ is not Σ_1, then there is some $\beta < \lambda_0^+$ such that $k(\beta)$ is ill-founded. Let us call the least such β, by analogy with Proposition 6.7.4, $\textsc{Std}(k)$. Since λ_1^+ is regular, we can therefore define

$$\alpha > \sup_{k \in \mathcal{E}^0_{I(\lambda_0)} - \mathcal{E}_{I(\lambda_0)}} \mathsf{j}(\textsc{Std}(k)) \tag{6.105}$$

By Lemma 6.8.2, let J be chosen such that for some $\eta < \delta_1$, $J_{0,\omega}(\eta) > \alpha$.

The remainder of the construction mirrors that of Claim 6.7.4.2 of Proposition 6.7.4. Namely, we will use J and its iterates to push the ill-foundedness of some $\mathsf{j}(k_{0,\omega})$ for some $k \in \mathcal{E}^0_{I(\lambda_0)} - \mathcal{E}_{I(\lambda_0)}$ below δ_1. The complication is that, in constructing π, we have access only to initial segments of k_0, k_1, k_2, and so on.

Ideally, for fixed k, one would again choose n_0 to be the least index such that $J_{0,n_0}(\eta) > j_0(\textsc{Crt}(k_0))$, then choose n_1 to be the least index such that $j_0(J_{0,n_1})(\eta) > j_0(\textsc{Crt}(k_1))$, and so forth, exactly as in Claim 6.7.4.2. Because π acts on elements of $\textsc{Seq}(\lambda_0)$, however, there is no *determinate* k from which we can deduce the $\langle n_i : i < \omega \rangle$.

Now, note that if $f = j^\alpha$ for some $j \in \mathcal{E}_{\lambda_0}$ and $\alpha < \lambda_0$, then we can still meaningfully speak of $\textsc{Crt}(f)$ and the iterates of f, with the caveat that if α is not large enough, then $\textsc{Crt}(f)$, $f(f)$, etc. may not be defined.

Now, we define the so-called *usable* sequences s in $\textsc{Seq}(V_{\lambda_0})$. To begin, if $|s| = 1$, then s is usable if $s(0) = f = k^{\alpha^0_i}$ for some $\alpha < 0$ so large that the cardinal $\textsc{Crt}(f)$

[7] That is, the central dichotomy holds: for any $x \in \overline{\textsc{Seq}}(\lambda_0)$, either $\pi(x)$ is ill-founded below δ_0, where δ_0 is as in Lemma 6.8.2, or $\pi(x)$ is strongly well-founded.

is defined, and for the least n_0 such that $J_{0,n_0}(\eta) > \mathfrak{j}(\textsc{Crt}(f))$, $J_{0,n_0}(\eta) < \alpha_i^1$. Let $\eta_0 = \eta$, $\eta_1 = J_{0,n_0}(\eta_0)$, and let $\eta_2 = \mathfrak{j}(f)(\eta_1)$; these ordinals are all well-defined by hypothesis. Now for the induction step, let $|s| = m$, and suppose $\eta_0, \eta_1, \ldots, \eta_{2m-2}$ as well as $n_0, \ldots, n_{m-1}$ have all been defined. Then, we say that s is usable if:

(i) The sequence $s\restriction_{m-1}$ is usable;

(ii) Letting $f = s(|s| - 1)$, the ordinal $\textsc{Crt}(f_{m-1})$ is defined, where f_{m-1} is the $(m-1)$-st iterate of f;

(iii) The function $f = k^{\alpha_i^0}$ for i so large that the smallest integer n_m such that $J_{n_{m-1},n_m}(\eta_{2m-2}) > \mathfrak{j}(\textsc{Crt}(f_{m-1}))$ satisfies $J_{n_{m-1},n_m}(\eta_{2m-2}) < \alpha_i^1$.

Then, let n_m be as above, and let $\eta_{2m-1} = J_{n_{m-1},n_m}(\eta_{2m-2})$ and $\eta_{2m} = \mathfrak{j}(f_m)(\eta_{2m-1})$.

We define $\pi : \textsc{Seq}(\lambda_0) \to \mathcal{E}_{\lambda_1}$ in the following way. Let $\pi(\emptyset) = J$. If s is not a usable sequence, then $\pi(s) = J_{|s|}$. If s is a usable sequence, let $k_*^0 = \pi(s\restriction_0)$, $k_*^1 = \pi(s\restriction_1), \ldots, k_*^{i-1} = \pi(s\restriction_{i-1})$, where $|s| = i$. Let $s(|s|) = f$ for some $f \in I(\lambda_0)$. Then, $\mathfrak{j}(f) \in I(\lambda_1)$, so there is some $k_i^* \in \mathcal{E}_{\lambda_1}$ such that k_i^* extends $\mathfrak{j}(f) \circ J_{n_{|s|-1},n_{|s|}}$, where the sequence $\langle n_m : m \leq |s|\rangle$ has been inductively defined as in the preceding paragraph. Then, simply set $\pi(s) = k_i^*$.

Now, let $x \in \overline{\textsc{Seq}}(\lambda_0)$. Then, note that there is some *single* $k \in \mathcal{E}_I$ such that $x = \langle k^{\alpha_{n_i}} : i < \omega\rangle$. Notice that if s is not a usable sequence and $s \subset t$, then t is not a usable sequence. Therefore, if $x\restriction_n$ is not usable for any n, then $x\restriction_m$ is not usable for $m > n$, and so it follows that $\pi(x)$ is strongly well-founded by construction. If $x\restriction_n$ *is* usable for all n, then set $k_*^n = \pi(x\restriction_n)$ and notice that by the definition of goodness,

$$k_*^0(\eta_0) = \big[\mathfrak{j}(k)\big](J_{0,n_0}(\eta_0)) \tag{6.106}$$

and, in general

$$k_*^i(\eta_{2i}) = \big[\mathfrak{j}(k_{0,i+1})\big](J_{0,n_i}(\eta_{2i})). \tag{6.107}$$

Hence $(\mathfrak{j}(k_{0,\omega}) \circ J_{0,\omega})(\eta_0) = (k_*)_{0,\omega}(\eta_0)$. Therefore, k is not a Σ_1-embedding, i.e., k is not iterable, if and only if $\pi(x)$ is ill-founded below δ.

(While the details are somewhat complicated, the heuristic is this: the usable sequences are those sequences for which one has enough information to ensure that the analogue of the construction in Claim 6.7.4.2 can be carried out. Thus, π is defined simply by carrying the construction out for these sequences, and producing a well-founded model otherwise.)

Proof of Theorem 6.8.1. Let $\mathbb{P} = \mathrm{col}(\omega, V_{\lambda_0})$, and let G be V-generic; then, in $V[G]$, π—modulo a bijection between ω and $\textsc{Seq}(\lambda_0)$—*actually* encodes a set of reals, in the sense of Definition 6.5.1. By a slight abuse of notation, we shall refer to this set as A_π.

Note that since $\mathrm{col}(\omega, V_{\lambda_0})$ is the set of all finite sequences in V_{λ_0} ordered by extension. Hence, for $r \in \mathbb{R}^{V[G]}$, there is a *canonical term* τ such that $\tau_G = r$, where τ is a set of triples p, n, m, $p \in \mathbb{P}$, $n, m \in \omega$ such that:

(i) If $(p, n, m) \in \tau$ then $(q, n, m) \in \tau$ for $q \supset p$;

(ii) For any $p \in \mathbb{P}$ and any $n \in \omega$, there exists $q \supset p$ and $m \in \omega$ such that $(q, n, m) \in \tau$;

(iii) If $(p, n, m) \in \tau$ and $(p, n, i) \in \tau$, then $m = 1$;

and where τ_G is the interpretation of τ by G.

Then, we have the following lemma.

Lemma 6.8.14. *Suppose that $|V_\lambda| = \lambda$ and G is V-generic for collapsing V_λ to ω. Suppose that there are infinitely many Woodin cardinals above λ and a measurable cardinal above them all. Suppose that in $V[G]$, $A \in P(\mathbb{R}) \cap L(\mathbb{R})$. Then in V, there exists $X \subset V_{\lambda+1}$ such that:*

(i) The set X is a canonical term for a set of reals;

(ii) The set $A = \{\tau_G : \tau \in X\}$, where τ_G is the interpretation of τ by G;

(iii) Both X and $(V_{\lambda+1} - X)$ are Σ_2-definable in $V_{\lambda+1}$ from parameters, i.e., X is Δ_2-definable in $V_{\lambda+1}$ from parameters.

Since the hypotheses of Lemma 6.8.14 are trivially satisfied by λ_0, suppose toward a contradiction that $A_\pi \in L(\mathbb{R})^{V[G]}$; then there is a canonical term $X \subset V_{\lambda+1}$ that is Δ_2 definable from parameters in $V_{\lambda+1}$ such that $X_G = -A_\pi$. But then, note that

$$\mathcal{E}_I = \{k \in \mathcal{E}_I^0 : \text{for all sequences } \langle i_n : n < \omega\rangle \text{ and } p \in \mathbb{P} \\ \text{such that } x = \langle k^{\alpha_{i_0}}, k^{\alpha_{i_1}}, k^{\alpha_{i_2}}, \ldots\rangle \in \overline{\text{SEQ}}(\lambda_0), \text{ and } p \not\Vdash \dot{\check{k}} \in X\} \quad (6.108)$$

and so $\mathcal{E}_I$ is Δ_2-definable, contrary to Lemma 6.8.11.

Therefore, $A_\pi \notin L(\mathbb{R})^{V[G]}$, and so by Lemma 6.8.3, $L(\mathbb{R}^{V[G]}) \cap \mathcal{P}(\mathbb{R})^{V[G]} \subset \Gamma^{V[G]}$, and since Γ is determined, **AD** holds in $L(\mathbb{R}^{V[G]})$. Hence, by Lemma 6.8.5, it follows that **AD** holds in $L(\mathbb{R})^V$. □

Definition 6.8.15 (Universally Baire Sets)**.** Call a set $A \subset \mathcal{N}$ *universally Baire* if, for every compact Hausdorff space X and any continuous map $f : X \to \mathcal{N}$, $f^{\text{Pre}}(A)$ has the property of Baire in X.

Remark 6.8.16. The universally Baire sets represent, in some sense, the ultimate generalization of the projective sets. While the results presented in this chapter are subsumed by those of Martin and Steel in [MS88], the techniques of Section 6.8 can be easily adapted to prove that every universally Baire set has an A-representation, and hence, is determined.

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